The Education Column

by

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Logic For High School Mathematics Teachers

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Abstract

This article addresses the question of whether concepts from formal logic can meaningfully contribute to mathematics education at the school level. We report on an attempt to work with high school teachers of mathematics in India on notions from logic such as: the notion of truth relative to a structure, construction of models for a set of sentences, consistency of procedures in algebra, and reasoning about algorithms used in school and comparing those algorithms systematically. Broadly, we tried to engage teachers in using ideas from logic to help with students' misconceptions, and to help students with reasoning about procedures.

1 Background

Our story begins at a workshop for high school teachers, the theme of which was *problem solving in algebra and geometry*. Typically, in such workshops teachers engage in problem solving, discuss strategies used in problem solving, and then move on to discussions on their use in classroom pedagogy.

In a particular session, there was a discussion on problem solving techniques that had come up in different contexts, and while we were listing them, one teacher V made the remarkable assertion that there was no unifying method combining these techniques, especially in the context of algebra. He contended that they were "tricks" and best learned as such. Some teachers agreed with him, while several others disputed this assertion. However, when called upon to present a unifying method, they found it hard to even argue that any such method exists, let alone present one, much to the glee of the (by now growing) camp of "algebra is a collection of tricks" enthusiasts.

At this point, M, another teacher, wondered whether this was a problem specific to algebra, and whether the situation was different with respect to geometric reasoning. She added that geometry was about proofs, and that should help. I turned to V and asked whether his geometry was also a bunch of tricks. V was emphatic that geometry was "logical," entirely based on axioms and proofs, and hence all the techniques used were unified into one whole. The ensuing discussion led to a consensus among teachers that the axiomatic method in geometry provided a sound basis for problem solving as well.

I intervened and reminded teachers that algebra was entirely axiomatic in its development. A teacher protested that this was true of "abstract algebra" studied at university, but not algebra in school mathematics which was only about "solving equations." This led to a general discussion on the place of school algebra within the broader study of algebra (at university), but it was clear that there was an uneasy realisation among the teachers that algebra could not be axiomatic and at the same time be a bunch of tricks.

School algebra is largely about setting up and solving systems of linear equations, and trying to solve quadratic and cubic equations; all this is carried out over the reals. I presented a proposition to teachers that a uniform algorithm, which could be implemented as a computer program, could solve any of these problems; I asked whether they believed this proposition to be true. An overwhelming subset of the teachers answered yes. When I asked them for reasons why they thought so, they broadly referred to computers being "powerful." One of the dissenters referred to *Gödel's Incompleteness Theorem*: she said that Gödel had shown that we could not even solve "general problems" involving "just arithmetic," so solving difficult cubic equations on the reals "simply could not be done."

The incident led to the realisation that while high school teachers of mathematics see reasoning as being fundamental to mathematics, they do not necessarily believe that there is logical structure in reasoning. Further, they do not perceive any connection between logic and algorithms or procedures. This realisation raises the following questions.

1. Is formal logic indeed necessary, or even helpful, in the teaching / learning of mathematics?

2. Mathematical learning at school involves learning a range of rules and procedures. Can logic help in seeing these as *inference systems* and provide teachers with structure (rather than seeing them as isolated tricks)?

While answers to these questions would eventually have an impact on the larger questions relating to mathematics school curricula and to helping children learn mathematics and logic, our effort was more modest. Our goal was to see whether mathematical logic could help to improve the teachers' own content knowledge and pedagogical content knowledge of mathematics. My own positive conviction in this regard was tempered by some doubt. A series of workshop opportunities for interacting with mathematics teachers led to a strengthening of this conviction, sharing which is the main aim of this article.

A caveat: this work was not exclusively focussed on logic. I have been working on mathematics school education for a long time and was working with high school teachers of mathematics anyway. The opportunity this provided for discussions about formal logic and introducing mathematical logic to them also led to the identification of a number of ways logic could help to clarify mathematical content to themselves, and to help them address student misconceptions more effectively. These are the insights I attempt to share in this article.

Some of the ideas here were presented in the context of a discussion on school curriculum published in 2023 [18].

2 Logic in School Mathematics

Despite logic and reasoning being considered central to mathematics and computing education, they play a largely peripheral role in high school or undergraduate curricula; at least, this is the case in India.

The content areas of high school mathematics have remained principally unchanged since the days of the industrial revolution: arithmetic, algebra, geometry, trigonometry, probability and statistics, and an introduction to differential and integral calculus. There are a few chapters on combinatorics and propositional logic, but they stand in isolation from the rest of the chapters. In terms of relative space, arithmetic in the early years, algebra and geometry in the middle years, and calculus in the later years occupy the maximum [16]. While there are differences across national curricula, the principal structure remains similar.

In what follows, we focus on the curriculum of the *Central Board of Secondary Education (CBSE)* in India, and that of the *Tamil Nadu State Board of Education* in India, essentially because of our familiarity with them and relative ignorance of syllabi elsewhere. However, even a superficial study of curricula across the world shows broad similarities to the Indian CBSE. In India, all the schools in a state (such as Tamil Nadu) affiliated with the state board follow the textbook of that board. In the case of Tamil Nadu, this constitutes nearly 75 % of the schools. Of the remaining, most are affiliated with the CBSE. Thus the CBSE textbook is used by a number of schools across the country from many states.[1]

While the focus of this article is on logic for school teachers of mathematics, and not on the mathematical content they teach, it is necessary to have an understanding of the way school the mathematics syllabus is structured and textbooks are designed,

[1] The story is more complex, as there are boards of education other than these, but this simplification suffices for our discussion.

in order to understand teacher beliefs about the role of logic in mathematics education.

In this curriculum, the study of logic is principally limited to *geometric reasoning* and *using the language of propositional logic*. We discuss these below.

2.1 Geometric Reasoning

Geometry occupies significant space in the secondary school curriculum, in grades 9 and 10. In grade 9, it constitutes nearly half of the instruction period (75 out of 160 scheduled lessons). In grade 10, it reduces to less than a quarter (30 out of 160). *Proofs* are accorded importance in the study of geometry. There are theorems and proofs in other areas such as number theory and algebra, but these are not as explicitly discussed as they are in geometry. Some algebraic identities are stated without proof, whereas geometric propositions are (almost) always proved.

Geometry begins with a historical introduction to Euclid. The syllabus document [14] states:

> Euclid's method of formalising observed phenomena into rigorous mathematics with definitions, common/obvious notions, axioms/postulates, and theorems. The five postulates of Euclid. Equivalent versions of the fifth postulate. Showing the relationship between axiom and theorem.

Subsequently a number of propositions on lines, angles, triangles, quadrilaterals, and circles are stated and proved. Geometry in grade 10 is similar, with the additional notions of congruence and similarity for triangles and tangents in the case of circles. There are also topics combining these, such as on cyclic quadrilaterals. The proofs are largely rigorous but informal. Some of the proofs, such as the ones using congruence, require some depth of reasoning using many assertions proved earlier. An appendix of each textbook (of grades 9 and 10) discusses the nature of deductive proofs, stressing the role of formal derivations and the distinction between verification and proof of statements.

In this regard, geometry instruction employs logic purposefully even while not stressing on the *language* of logic or on formal deduction in an inference system.

2.2 Propositional Reasoning

The CBSE syllabus for grade 11 includes a unit titled *Mathematical Reasoning*. The document [14] states:

> Mathematically acceptable statements. Connecting words / phrases – consolidating the understanding of "if and only if (necessary and

sufficient) condition", "implies", "and/or", "implied by", "and", "or", "there exists" and their use through a variety of examples related to real life and Mathematics. Validating the statements involving connecting words – difference between contradiction, converse, and contrapositive.

The textbook chapter introduces the language of propositional logic, and explains the use of connectives very well. The principal focus is on implication: getting students to appreciate the distinction between $p \implies q$ and $q \implies p$, to understand inclusive disjunction and to realise that a single counterexample suffices to falsify a universally quantified statement. Considerable effort is spent in formalising intuitive statements in the language of propositional logic. It is also important to point out that geometry is extensively used as the terrain from which propositions are picked.

For instance, an exercise [15, Chapter 14, page 344] asks the students to *write the following statement in five different ways, conveying the same meaning*.

If a triangle is equiangular, then it is an obtuse angled triangle.

For someone formalising this as $p \implies q$ (which is what the book teaches the student to do), there are indeed equivalent forms to try, such as $\neg q \implies \neg p$. If a student tries to use the language of geometry starting with a triangle ABC and proceeding further, it is much less clear if they would get anywhere with this exercise. (Equating differently expressed properties by virtue of logical equivalence is an important logical exercise itself. Our remark here is not to downplay this, but to point to the limited use of formalisation.)

2.3 Teacher Beliefs Related to Logic

Given that this is the logical content that teachers are expected to teach, it is not surprising that they consider geometry as the central location of logic in mathematics learning. They see that logic is essentially about seeing how the theorems proved are deduced from clearly stated axioms. Logic is not associated with discovery but with writing in the formal language of mathematics. Indeed, during later discussions, some teachers asserted that logic was a kind of mathematical bureaucracy, arising only out of an insistence on stating things in a strictly formal (and unfriendly) way. This is despite the textbook being relatively informal: in reality, the axioms and inference rules of geometry are never formally spelt out, an informal argument is preferred over a formal deduction in the textbook.

Indeed, the task of reasoning and proving is important in the mathematics classroom, and this is best done in an informal intuitive way, rather than by insisting on formal deductions. There is extensive literature in mathematics education research on students' reasoning and structure in proofs [9, 11, 19]. Invariably, these researchers point out that developing *intuition* is more important for mathematics learning than formal proofs. So it is not surprising that mathematics teachers are of similar opinions.

We asked teachers whether the language of logic plays any significant role in teaching geometry, or whether we teach deductive systems by way of geometry. Their answers to both of these questions were largely negative. Some authors argue that mathematical logic is not necessary at all for teaching proofs in mathematics [1], and some education researchers offer nuanced arguments on specific aspects of logic being relevant in the teaching of proofs [6].

On the other hand, learning the correct use of propositional connectives and quantifiers unambiguously constitutes logic learning, and this is also indispensable for obtaining fluency in the language of mathematics. However, this is also compromised in classroom practice [10]. Indeed, in our conversations with teachers, many wondered what the use of all this "Boolean logic" was, when they had to teach difficult concepts in differential and integral calculus.

To understand why they have such a reaction to propositional logic, it may be instructive to look at the *placement* of the logic syllabus unit in the curricular structure. The other syllabus units among which Boolean logic is placed involve topics such as Trigonometric Functions, Complex Numbers, the Binomial Theorem, Conic Sections, Frequency Distributions, Limits, and Derivatives. In fact, in the CBSE's textbook, the chapter on propositional logic follows the chapter on limits and derivatives. After solving exercises computing the derivatives of functions such as $x/(\sin x)$, the students encounter the relationship between $p \implies q$ and $q \implies p$. It is hardly surprising that they are left wondering whether this is merely an interlude between differentiation and integration. While students are happy to take any "easy" topic that comes their way, the deeper issues pass them by, and the chapter on propositional reasoning merely becomes another set of (thankfully simple) rules to be learned and forgotten soon. Teachers are broadly in sympathy with this viewpoint as well.

If teachers see very little use for logic in school mathematics, and even see the little that is on offer as being ineffective, should we conclude that logic is irrelevant to school mathematics?

This initiated a different conversation between us: rather than seeing logic as a topic in mathematics to be taught to students, should we explore whether there were demands from the teaching of algebra, geometry, number systems, and so on, that logic could address? This required us to go a little deeper into logic. What followed was a formal introduction for the teachers on first order logic, interpreted over the integers and the reals.

3 The Content

In this section, we describe our efforts in trying to come up with an appropriate *Logic Curriculum* for mathematics teachers. The idea was not to aim for "direct impact" on the system in the sense of a curricular reform or textbook writing, but an indirect one, by enriching teachers' understanding of logic so that they could use it in pedagogic contexts offered by the curriculum and textbook.

The initial ambition was to devise a systematic course for teachers, one that would eventually be approved by the state education machinery. But conducting a course of any significant duration for teachers was difficult for a variety of reasons, the principal one being teachers' availability, and the other being official permissions. Instead, what we did was to incorporate logic sessions in a series of teacher workshops on mathematics. This meant that some of the content had to be repeated, and also that it was not the same set of teachers present in all the workshops. However, there was a core of more than 10 teachers present through all engagements who provided continuity and helped maintain coherence to the effort.

We also need to point out that in India, elementary school teachers have a degree in education but they do not necessarily have a disciplinary specialization. Teachers at the secondary stage are qualified in both education and mathematics. At the senior secondary stage, teachers come with a Master's degree in mathematics. In general, most undergraduate mathematics curricula in India do not include a course on mathematical logic, but incorporate some logic into a foundational course and some set theory into a course on real analysis. Again, when students are grappling with completeness of ordered fields, logic tends to take the backseat. As a result, very few teachers have learned logic formally, nor is it in any way emphasized in their education courses (which center on pedagogy rather than content). In particular, *none* of the teachers in our group had learned mathematical logic formally.

The first step was to identify a logical language in which much of school mathematics can be expressed. First order logic with addition and multiplication (later extended with exponentiation) works for most of algebra, geometry, and number theory in school. At the senior secondary level, when trigonometric functions, limits, and differential calculus are studied, this is inadequate, but our discussions were mainly with high school teachers.

Setting up a language and presenting its syntax and semantics was largely new to teachers. We considered the same sentences interpreted over whole numbers, over integers, and over reals; and the differences were greatly appreciated by teachers. We also saw equations as sentences when quantified appropriately, and steps in solving equations as implications between sentences. What perhaps caught the attention of teachers most was the possibility of algorithms that could check whether an equation had a solution or not. It was also a surprise to them that it was possible to design algorithms over the reals but not the integers. (Admittedly, these theorems were only stated without proof; however, some understanding was possible mainly because the teachers had the language for it.) We also discussed the idea of correctness proofs of algorithms, with the Euclidean algorithm being a good example. We had hoped to show this proof by formalising it in an interactive theorem proving tool, but this turned out to be too difficult in the time we had.

4 Students' Difficulties

This series of interactions led to teachers identifying a range of mathematical difficulties faced by students where logic might be of significant help. In each of the cases, the teachers felt that they could offer better explanations after their own understanding of mathematical logic. While these difficulties are acknowledged and discussed extensively in the mathematics education literature, what was perhaps novel and interesting in our discussions was in the suggested use of mathematical logic for addressing these difficulties.

1. **The use of variables:** We ask middle school students to solve equations of the form $x + 3 = 5$, and they learn that x is a specific unknown number. Then they go on to consider equations such as $x + y = 5$, when x can be one of many numbers, somehow dependent on the value that y takes. We also ask them to "see" that $x + y = y + x$, but now x can be *any* number whatsoever. There is worse to come when we ask them to consider the line given by the equation $x = k$, where k is some constant [3]. All these examples are legitimate and reasonable uses of x in mathematical contexts, but being syntactically disciplined about quantifying x appropriately depending on the context can solve much of the confusion. For teachers, it was a matter of realizing that we could instead work with sentences such as "Is there an x such that $x + 3 = 5$?" and "Given *any* y, can you find *some* x such that $x + y = 5$?", etc. This can be followed up with the question "Can you find more than one such x?" and then go on to discussing an *expression* for y in terms of x.

2. **True or false:** Durand-Guerrier [7] discusses the proposition "Any number that ends in 4 is divisible by 4." Some students consider such statements true as well as false since they hold for some numbers and do not hold for some other numbers. I have once come across a student who, when asked whether $x < y$ implies $x^2 < y^2$, answered that it was half-true since it was true for exactly half the numbers. Without getting quite philosophical, it is possible to discuss the notion of a mathematical sentence being true or

false, depending on the restrictions we can place on the values taken by the variables occurring in the sentence.

3. **Truth in a structure:** Is $x^2 - 2 = 0$? Or more precisely, does that equation have a solution? Well, that depends on which number system you are solving the equation in. But this is a source of endless confusion to students.

   The problem is not only that of truth being structure-dependent, but of interpretation of operations and functions being structure-dependent as well. This confusion gets considerably worse when we consider statements such as "Multiplication is the same as repeated addition." This works on positive integers and it is how students learn arithmetic in primary school. But this reasoning is unhelpful when they need to learn $\sqrt{2} \cdot \sqrt{3}$, and so on. This is where the syntax–semantics distinction taught in logic proves to be very helpful.

4. **Equation solving:** A classic problem asks students to evaluate $(x^2-1)/(x-1)$ at $x = 1$. The student answering that the function evaluates to 2 gets shocked when made to realize that the function cannot be evaluated at $x = 1$. The same student, further on, is asked to compute the limit of $(x^2 - 1)/(x - 1)$ as $x \to 1$. Now the limit is indeed 2. The student is left with a bad taste in their mouth: something is surely wrong!

   There are several issues here. Typically the equations, when interpreted as equating functions, require both sides to be defined over the same domain. In the case of limits, the equation is over numbers. The inference system at work in both the cases is not the same, but we do not distinguish them.

5. **Heuristic advice**: One of the side effects of the trouble above is the typical advice given to students to always substitute the answer they obtain in the given equation and verify. This does help in the instance of function evaluation above, since the student can immediately verify that the substitution $x = 1$ leads to division by zero.

   Equation solving comes with a range of heuristics such as grouping terms, moving all terms involving the same variable to one side, changing the sign across the equality symbol, and so on. However, very rarely does a student get any assurance on the reliability of these heuristics, and whether they suffice in all contexts. The answer to this cannot be given without studying the underlying inference systems on equational reasoning.

6. **Functional variation:** The student learns the definition of the *sine* function as the ratio of the 'opposite side' to the hypotenuse of a right-angled triangle. Later on, the student gets to graph the function as an oscillating curve. What

is varying here? Are we saying something about the universe of right-angled triangles? This again is a cause of confusion for many students. It is a much harder question to address, and the limited first order reasoning we discussed was not sufficient to articulate an explanation.

7. **Proving vs. checking:** Typically, proofs in school mathematics involve demonstrations of the kind that some universally quantified statement is true in some implicit structure. The quantification could be over numbers, triangles, circles, and so on. (Note that even the assertion of an infinity of primes is of the form $\forall x \exists y$.) On the other hand, students are used to checking that a property holds for some object: for instance, that 137 is a prime, or that the perpendicular bisectors of a given triangle have a point of concurrency, and so on. Yet the former isn't called a proof whereas the latter is termed a *theorem*. These are clearly logical issues, and they can easily be cleared up with some understanding of logic.

There are many such sources of confusion and incomprehension in school mathematics. We mention these here to point out that teachers could see logic as being helpful towards addressing them.

5 Reasoning About Algorithms

What follows is a different discussion, one that we had with *middle school teachers*, where we did not discuss first order logic, but the use of logic in a more informal sense, emphasizing reasoning rather than logic.

If there is one thing that characterizes school mathematics for a student, it is the learning of algorithms in one context after another; be it long division, factorization of quadratic expressions, matrix inversion, computing compound interest, or finding the standard deviation of given data, and so on. These are largely seen to be disparate, to be learned and applied in context. It is not an exaggeration to say that the teaching and learning of these algorithms dominates school mathematics almost to the exclusion of their declarative content. Moreover, the conceptual vs. procedural debate dominates the discourse on mathematics education.

We mention algorithms to point out that *reasoning* about algorithms is crucial for bridging the conceptual–procedural divide (where it exists), and that this is an indispensable role for logic. Moreover, it is such reasoning that is central to **computational thinking**, a term of great current interest. Indeed it is when algorithms are taught and learned as procedures without any explicit reasoning about them that mathematics becomes difficult for many. The position paper on teaching of mathematics in the *2005 (Indian) National Curriculum Framework* [16] asserts:

> . . . we have repeatedly referred to offering a multiplicity of approaches, procedures, solutions. We see this as crucial for liberating school mathematics from the tyranny of the one right answer, found by applying the one algorithm taught. When many ways are available, one can compare them, decide which is appropriate when, and in the process gain insight.

When we have a multiplicity of methods, they need to be compared and analyzed to determine which one works best when, and this is the conceptual understanding of importance to mathematics learning. Ideally when students come up with algorithms, compare theirs with other algorithms and argue about them, they acquire confidence in the use of these algorithms. Logic has a great deal to contribute in this regard.

Opportunities for such reasoning are present throughout school. We can already see this with very small children, less than 5 years old: the child is given (say) 20 coins, plays with them, and is then asked to find all the 20 coins and carefully put them in a box. When asked "Have you found all the coins?", they learn to count and make sure. Further, when again asked "How do you know you have counted all the coins?" initially they re-count, but later come up with grouping strategies so that verification is easier. When a child who has distributed 20 toffees among 5 children is asked "How many do you have?" she answers readily. When asked further "How many do each of your friends have?" she needs to reason explicitly by symmetry, assuming that the algorithm she used for toffee distribution treated all children symmetrically. Typically she would have given one toffee to each child in a full round, repeating this for four rounds.

At primary school, students get asked how many different pairs of whole numbers add to (say) 10. When a student offers the solution, there is an implicit question to her: how do you know you have counted them all? This process of verification involves reasoning, and when it becomes a habit, shows the value of formal reasoning as it helps the student to catch routine mistakes. The transition from "do you know" to "how do you know it" is important for mathematics, and reasoning provides the natural vehicle for carrying the student through the transition.

One can list a range of algorithms in number theory, algebra, geometry, trigonometry, calculus, statistics, and business mathematics and ask, in each case, how do students reason about the correctness of the algorithm they learn. Much better, if and when they get to devise algorithms of their own, one can ask how they argue that their method works. While formal proofs of correctness may neither be feasible nor required, argumentation is important, and logic can help in this.

All this, of course, is even more relevant to computing education. Reasoning about procedures is at the heart of computational thinking [17], and the connection to logic is generally missed in school. When a small child is asked to add 2, 104, and 78, and regroups them to first get 80 and then adds it to 104, this re-ordering is a crucial element of computational thinking. Over time a student builds up such rules, and reflection on them and their applicability is a logical exercise, very much necessary for meta-cognition.

Students learning programming need to also learn to build correctness arguments (even if not proofs). Reasoning about programs requires understanding the syntax–semantics distinction, and checking whether a claim is true or not at different points of program execution. Logic plays an indispensable role here [8].

In our interactions, teachers could come up with multiple learning tasks that would encourage children to informally argue the correctness of algorithms used in class. We did not discuss formalisation of these arguments very much.

6 In Conclusion

Experiential accounts of educational interventions are of very limited value. What we need is research based on strong data from classroom practices, analyzed in appropriate theoretical frameworks. Our discussion here should be seen as stressing the need for such research, one that examines whether a logician's perspective may be able to influence mathematics and computing education at the school level positively; and if so, understand how, in a nuanced manner.

As a rule, mathematics educators seem to be largely unaware of the interaction between mathematical logic and school mathematics, and what logic might mean at different stages of schooling. While there is extensive literature on how students reason in mathematics classes, this is not examined in the light of logic in itself.

The aforementioned position paper on teaching of mathematics in the *2005 (Indian) National Curriculum Framework* [16] talks about *compartmentalisation* as one of the major problems of the mathematics education milieu, with little interaction between mathematics teachers at university and those in high school; or between the latter and teachers in the elementary school. Conversations between high school teachers and university teachers involved in formal methods in computing can perhaps be of benefit to both.

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