The Formal Language Theory Column

by

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Generalized Directions on a Compass

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Abstract

Beside the four cardinal directions on a compass (North, East, South, West), there are more fine grained directions between them, such as NorthEast and EastNorthEast. They are formed by concatenating neighboring directions, but not arbitrarily: EastNorth and NorthEastEast are incorrect directions. We study the underlying (though not explicitly given) recursive naming procedure. As it turns out, these generalized directions on a compass are indeed unique and are formed by a tabled Lindenmayer system, usually used to describe cellular plant growth. As we show, deciding whether a given string actually forms a direction is solvable in linear time.

1 Introduction

A typical Western world compass rose contains, in clock-wise order, the four cardinal directions North (N), East (E), South(S), West(W).[1] All further directions are a combination of these four words. We consider the naming of those between N and E only, as all other other directions are formed symmetrically.

As the basic principle, the direction exactly half-way between, say N and E, is formed by fusing these two names together, in this case to NE but not to EN. The preference for NE over EN on this level is somewhat arbitrary, but the recursively formed names follow a pattern: The direction between N and NE is not ENN but is NNE. Similarly, the direction between E and NE is ENE, and not NEE. The underlying principle is to form, say NNE, is to put N before NE since N compared to NE has been derived in fewer steps. Hence also E and NE combine to ENE. Following this pattern, NE and NNE combine to the direction NENNE, and so on. (Section 2 will state the precise definition. See also Figure 1 and Figure 2.)

This is arguably the natural (recursive) *linguistic* interpretation of how directions on a compass are formed. In reality, though, this principle is soon abandoned in favor of a *numeric* measurement in degrees (0° for N, 90° for E and so on).

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[1] Historically and internationally, there are other forms of compass roses, see [3, 9].

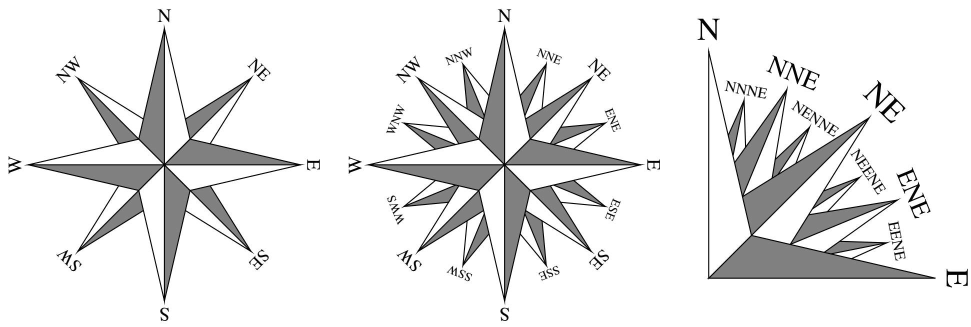

Figure 1: (Part of) compass roses, with ever more refined directions

Such a numeric measurement is easy to use, and, on the surface, this might be the reason to favor them over the seemingly complex linguistic interpretation. However, strictly algorithmically speaking, the linguistic interpretation is also easy to use. As we show, the recursive naming procedure produces unique names for the directions on the compass, and further, deciding whether a given string actually represents a direction on a compass (and if yes, which one) is decidable in linear time, i.e., deciding the following problem:

Input: A word $w \in \{\mathsf{N}, \mathsf{E}\}^*$ of length n.
Question: Is w a direction on the compass strictly between N and E?

Further, there is a strong connection to rewriting systems. As we show, the generalized directions on a compass can be defined equivalently by a *context-free tabled Lindenmayer system* (tabled 0L-system). Here 'context-free' refers to the rules of the L-system. (In fact, the direction on the compass do *not* form a context-free language!) L-systems were introduced by Lindenmayer in 1968 to describe cellular plant growth [6]. The properties and expressiveness of different kinds of L-systems were extensively studied, see for example the book chapter of Kari et al. [5]. L-systems have been tied to natural languages before. Becerra-Bonache et al. [2] point to that certain aspects of natural language are not context-free and propose that L-systems pose a better model for natural languages.

Section 2 properly defines the directions on a compass. Section 3 introduces the theory then used in Section 4 to algorithmically detect directions. Finally, Section 5 points to the connection to rewriting systems.

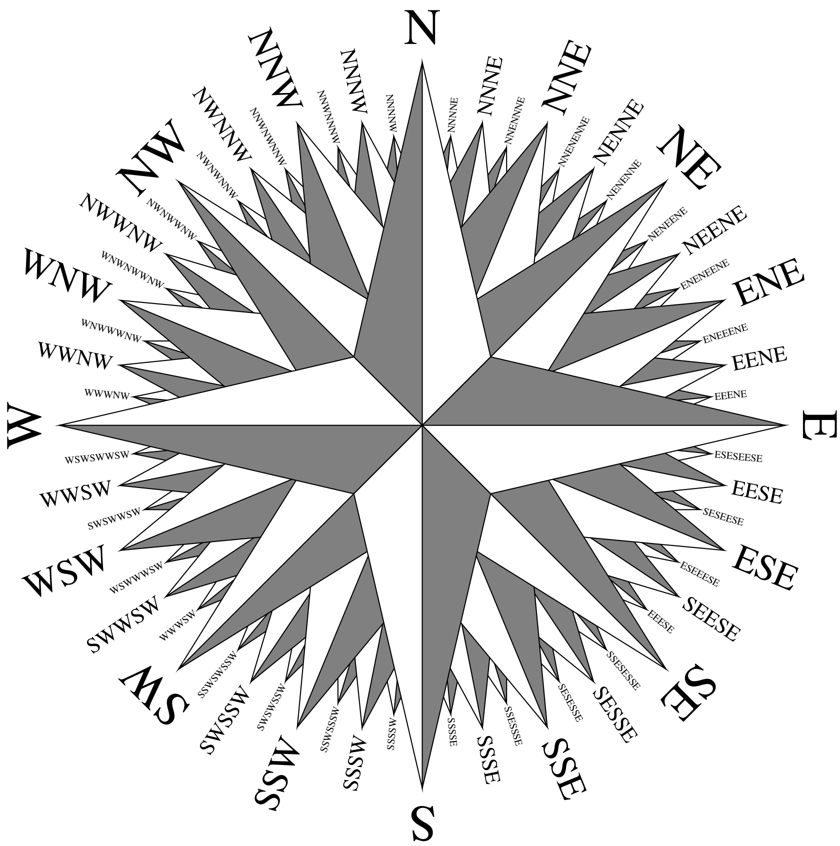

Figure 2: A detailed compass rose

2 Two Definitions

By symmetry, we may only consider the directions strictly between N and E. There is a *mechanical* and an *elegant* way to define the set of directions $\mathcal{D}$. The *mechanical* way defines $\mathcal{D}$ as the union over $i = 1, 2, \ldots$ of the leaves of the ordered rooted binary tree T_i, defined as follows:

1. Tree T_1 only contains the root N with a right child E and NE as the left child of E. (That is, T_1 has in-order traversal N, NE, E and NE is the only leaf).

2. For $i \geq 1$, let L_i be the in-order traversal of the leaves of T_i. Then tree T_{i+1} results from T_i by adding for every leaf v with predecessor ℓ and successor r in L_i a left child ℓv and a right child rv. (For example, T_2 has in-order traversal N, NNE, NE, ENE, E and leaves NNE, ENE. See also Figure 3.)

The directions ℓv and rv obtained in above step 2. are located exactly half-way between ℓ and v, respectively between r and v. Hence the in-order traversal is the order in which the directions appear clock-wise on the compass.

The *elegant* way defines $\mathcal{D}$ by using the following set $\mathcal{D}^-$. Set $\mathcal{D}^-$ contains each word of $\mathcal{D}$ but with a hyphen at a suitable position. Instead of $\mathsf{NE} \in \mathcal{D}$ we have $\mathsf{N}\text{-}\mathsf{E} \in \mathcal{D}^-$. Instead of a direction $\ell v \in \mathcal{D}$ and $rv \in \mathcal{D}$ formed in step 2 we have $\ell\text{-}v \in \mathcal{D}^-$ and $r\text{-}v \in \mathcal{D}^-$, respectively. For example $\mathsf{NNE} \in \mathcal{D}$ becomes $\mathsf{N}\text{-}\mathsf{NE} \in \mathcal{D}^-$ (where the hyphen also indicates a pause while speaking).

Consider i such that tree T_i has leaf $uv \in \mathcal{D}$ (corresponding to $u\text{-}v \in \mathcal{D}^-$). Then v is the parent of uv and u is another ancestor of uv. More so, we know the two *children* of uv in the tree T_{i+1}. Indeed, the in-order traversal L_i contains the subsequence u, uv, v or its inversion. That is, we can define $\mathcal{D}^-$ inductively by three simple rules:

1. $\mathsf{N}\text{-}\mathsf{E} \in \mathcal{D}^-$,
2. $f_{\mathsf{N}}(u\text{-}v) \in \mathcal{D}^-$ if $u\text{-}v \in \mathcal{D}^-$ for some $u, v \in \{\mathsf{N}, \mathsf{E}\}^*$, and
3. $f_{\mathsf{E}}(u\text{-}v) \in \mathcal{D}^-$ if $u\text{-}v \in \mathcal{D}^-$ for some $u, v \in \{\mathsf{N}, \mathsf{E}\}^*$; where

the mappings $f_{\mathsf{N}}, f_{\mathsf{E}}$ are defined for inputs $u\text{-}v$ with $u, v \in \{\mathsf{N}, \mathsf{E}\}^*$ as follows:

$$f_{\mathsf{N}} : u\text{-}v \mapsto u\text{-}uv,$$
$$f_{\mathsf{E}} : u\text{-}v \mapsto v\text{-}uv.$$

See Figure 3 for the derivation tree of $\mathsf{N}\text{-}\mathsf{E}$ using mappings $f_{\mathsf{N}}, f_{\mathsf{E}}$ at most three times. Let ρ be the mapping that removes any symbol '-' from its input word. Then $\mathcal{D}$ is the set of words $\rho(w)$ with $w \in \mathcal{D}^-$. For example, $\mathsf{NNENENNE} \in \mathcal{D}$, because it can be derived from $\mathsf{N}\text{-}\mathsf{E}$ by repeatedly applying f_{N} and f_{E}, and finally ρ:

$$\mathsf{N}\text{-}\mathsf{E} \overset{f_{\mathsf{N}}}{\to} \mathsf{N}\text{-}\mathsf{NE} \overset{f_{\mathsf{E}}}{\to} \mathsf{NE}\text{-}\mathsf{NNE} \overset{f_{\mathsf{E}}}{\to} \mathsf{NNE}\text{-}\mathsf{NENNE} \overset{\rho}{\to} \mathsf{NNENENNE}.$$

The exact position in the tree (and hence also the position on the compass) is revealed by the sequence in which f_{N} and f_{E} are applied. That is, whether a direction uuv (and similarly vuv) is the left or the right child of uv depends on whether the in-order traversal of T_i has subsequence u, uv, v or its inversion (where i is such that T_i has leaf uv). One can observe that whether its the inversion or not, flips with every use of f_{E}.

We will work with the elegant definition hereafter.

3 Detecting Directions

Essentially we are tasked to repeatedly *undo* the mappings f_{N} and f_{E} but without knowing the position of the hyphen. That is, we would like to compute *inverse*

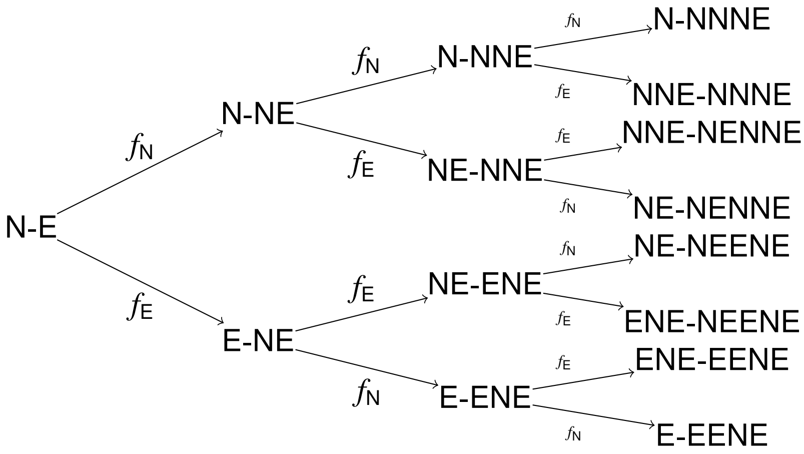

Figure 3: All derivations of N-E by applying $f_{\mathsf{N}}, f_{\mathsf{E}}$ at most three times. Ignoring the hyphens and edge labels, this is also the subtree of T_4 rooted at NE.

mappings f_{N}^{-1} and f_{E}^{-1}, assuming they are unique. We do so by observing an alternative way to derive the directions on a compass. This alternative derivation turns out to be easy to undo.

We rely on functions h_{N} and h_{E} that modify the current word *locally*. More precisely, h_{N} and h_{E} are homomorphisms and their replacement for each type of character is defined as: For $\mathsf{X} \in \{\mathsf{N}, \mathsf{E}\}$,

$$h_{\mathsf{X}} : \mathsf{N} \mapsto \mathsf{X}, \quad h_{\mathsf{X}} : \mathsf{E} \mapsto \mathsf{NE}, \quad h_{\mathsf{X}} : \text{-} \mapsto \text{-}.$$

That is, the character 'E' is always replaced by 'NE', while 'N' is replaced by 'N' or 'E' depending on the subscript of h. Additionally, h_{N} and h_{E} simply ignore the separator '-'. Then, surprisingly, h_{N} and h_{E} provide an alternative way to derive the directions, for example NNE-NENNE:

$$\text{N-E} \overset{h_{\mathsf{E}}}{\to} \text{E-NE} \overset{h_{\mathsf{E}}}{\to} \text{NE-ENE} \overset{h_{\mathsf{N}}}{\to} \text{NNE-NENNE}.$$

The sequence of the applied functions h_{N} and h_{E} resembles the *reverse* order in which f_{N} and f_{E} were applied. Hence we would like to to decompose, for example NE-NENNE, as

$$\text{NE-NENNE} \overset{h_{\mathsf{N}}^{-1}}{\to} \text{NE-ENE} \overset{h_{\mathsf{E}}^{-1}}{\to} \text{E-NE} \overset{h_{\mathsf{E}}^{-1}}{\to} \text{N-E}.$$

Then the order $h_{\mathsf{N}}^{-1} h_{\mathsf{E}}^{-1} h_{\mathsf{E}}^{-1}$ unveils the original order $f_{\mathsf{E}}, f_{\mathsf{E}}, f_{\mathsf{N}}$. This is no coincidence, as we will show.

3.1 The Inverses

It turns out that the inverses of h_{N} and h_{E} can be computed by parsing the given word w from left to right with very little lookahead. If w has prefix E, then it must be derived from a word with prefix N and using h_{E}. If it has prefix NN, it must be derived from a word with prefix N and using h_{N}. If it has prefix NE, it must be derived from a word with prefix E, and either h_{N} or h_{E} was used. The remaining part of w can be decomposed recursively. We thus can state explicit rules to compute h_{N}^{-1} and h_{E}^{-1}. For every suffix $v \in \{\mathsf{N}, \mathsf{E}, \text{-}\}^*$ and every $\mathsf{X} \in \{\mathsf{N}, \mathsf{E}\}$,

1. $h_{\mathsf{E}}^{-1}(\mathsf{E}v) = \mathsf{N}h_{\mathsf{E}}^{-1}(v)$, and $h_{\mathsf{N}}^{-1}(\mathsf{E}v)$ is not defined;
2. $h_{\mathsf{N}}^{-1}(\mathsf{NN}v) = \mathsf{N}h_{\mathsf{N}}^{-1}(\mathsf{N}v)$, and $h_{\mathsf{E}}^{-1}(\mathsf{NN}v)$ is not defined;
3. $h_{\mathsf{X}}^{-1}(\mathsf{NE}v) = \mathsf{E}h_{\mathsf{X}}^{-1}(v)$;
4. $h_{\mathsf{N}}^{-1}(\mathsf{N}) = \mathsf{N}$, and $h_{\mathsf{E}}^{-1}(\mathsf{N})$ is not defined;
5. $h_{\mathsf{N}}^{-1}(\mathsf{N}\text{-}v) = \mathsf{N}h_{\mathsf{N}}^{-1}(\text{-}v)$, and $h_{\mathsf{E}}^{-1}(\mathsf{N}\text{-}v)$ is not defined;
6. $h_{\mathsf{X}}^{-1}(\text{-}v) = \text{-}h_{\mathsf{X}}^{-1}(v)$; and
7. $h_{\mathsf{X}}(\varepsilon) = \varepsilon$.

We note that whether we consider directions with hyphen or not does not influence $h_{\mathsf{N}}^{-1}, h_{\mathsf{E}}^{-1}$. Recall that ρ is the mapping that removes any occurrence of '-' from the input word.

Lemma 3.1. *Let $w = u\text{-}v \in \mathcal{D}^- \setminus \{\mathsf{N\text{-}E}\}$ and $\mathsf{X} \in \{\mathsf{N}, \mathsf{E}\}$. Then $\rho(h_{\mathsf{X}}^{-1}(w)) = h_{\mathsf{X}}^{-1}(\rho(w))$. Further, $h_{\mathsf{X}}^{-1}(u\text{-}v) = h_{\mathsf{X}}^{-1}(u)\text{-}h_{\mathsf{X}}^{-1}(v)$ and $h_{\mathsf{X}}^{-1}(uv) = h_{\mathsf{X}}^{-1}(u)h_{\mathsf{X}}^{-1}(v)$.*

Proof. For inputs without infix $\mathsf{N\text{-}E}$, above rules 1.-7. essentially ignore the symbol '-'. That is $\rho(h_{\mathsf{X}}^{-1}(w)) = h_{\mathsf{X}}^{-1}(\rho(w))$ for every $\mathsf{X} \in \{\mathsf{N}, \mathsf{E}\}$ and word $w \in \{\mathsf{N}, \mathsf{E}, \text{-}\}^*$ without infix $\mathsf{N\text{-}E}$. In fact, this observation applies to $w \in \mathcal{D}^- \setminus \{\mathsf{N\text{-}E}\}$. That is, we claim that no word $w \in \mathcal{D}^- \setminus \{\mathsf{N\text{-}E}\}$ has infix $\mathsf{N\text{-}E}$. Recall that w has form $\ell\text{-}v$ or $r\text{-}v$ where ℓ, v, r is a subsequence in the in-order traversal L_i of the tree T_i with leaf v. Hence ℓ or r has suffix N. However, inductively we observe that no word in L_i but N itself has suffix N. Indeed, in L_1 only N itself has suffix N. By induction on i, in L_i again only N has suffix N since all new direction but $\mathsf{N\text{-}N}^{i-1}\mathsf{E}$ are formed by a concatenation words with suffix E.

Thus $\rho(h_{\mathsf{X}}^{-1}(w)) = h_{\mathsf{X}}^{-1}(\rho(w))$ for $w = u\text{-}v \in \mathcal{D}^- \setminus \{\mathsf{N\text{-}E}\}$ and $\mathsf{X} \in \{\mathsf{N}, \mathsf{E}\}$. Particularly, the rules of 1.-7. are applied to u and v independently, such that $h_{\mathsf{X}}^{-1}(u\text{-}v) = h_{\mathsf{X}}^{-1}(u)\text{-}h_{\mathsf{X}}^{-1}(v)$. Finally, we observe that $h_{\mathsf{X}}^{-1}(uv) = h_{\mathsf{X}}^{-1}(\rho(u\text{-}v)) = \rho(h_{\mathsf{X}}^{-1}(u\text{-}v)) = \rho(h_{\mathsf{X}}^{-1}(u)\text{-}h_{\mathsf{X}}^{-1}(v)) = h_{\mathsf{X}}^{-1}(u)h_{\mathsf{X}}^{-1}(v)$. □

Lemma 3.2. *For $w \in \mathcal{D} \setminus \{\mathsf{NE}\}$, at most one of h_{N}^{-1} and h_{E}^{-1} is defined.*

Proof. For the directions in L_2, NNE and ENE, only h_{N}^{-1} and only h_{E}^{-1}, respectively, is defined. By induction, any further word derived from ENE (in terms of f_{N} and f_{E}) contains the infix EE, hence only h_{E}^{-1} is defined. Similarly, any further word derived from NNE (in terms of f_{N} and f_{E}) contains the infix NN, hence only h_{N}^{-1} is defined. □

3.2 Uniqueness of the Directions

For $\mathsf{N\text{-}E}$, the mapping h_{N}^{-1} exactly undoes f_{N}, and h_{E}^{-1} exactly undoes f_{E}.

Observation 3.3. *For $\mathsf{X}, \mathsf{Y} \in \{\mathsf{N}, \mathsf{E}\}$, we have $h_{\mathsf{X}}^{-1}(f_{\mathsf{Y}}(\mathsf{N\text{-}E})) = \mathsf{N\text{-}E}$ if $\mathsf{X} = \mathsf{Y}$, and $h_{\mathsf{X}}^{-1}(f_{\mathsf{Y}}(\mathsf{N\text{-}E}))$ is undefined if $\mathsf{X} \neq \mathsf{Y}$.*

Proof. We have $h_{\mathsf{X}}^{-1}(f_{\mathsf{X}}(\mathsf{N\text{-}E})) = h_{\mathsf{X}}^{-1}(\mathsf{X\text{-}NE}) = \mathsf{N\text{-}E}$ for $\mathsf{X} \in \{\mathsf{N}, \mathsf{E}\}$, but $h_{\mathsf{N}}^{-1}(f_{\mathsf{E}}(\mathsf{N\text{-}E})) = h_{\mathsf{N}}^{-1}(\mathsf{E\text{-}NE})$ nor $h_{\mathsf{E}}^{-1}(f_{\mathsf{N}}(\mathsf{N\text{-}E})) = h_{\mathsf{E}}^{-1}(\mathsf{N\text{-}NE})$ is defined. □

Every direction in $\mathcal{D}^-$ other than $\mathsf{N\text{-}E}$ is derived from $\mathsf{N\text{-}E}$. The key insight is that h_{N}^{-1} and h_{E}^{-1} commute with f_{N} and f_{E}.

Observation 3.4. *Let $\mathsf{X}, \mathsf{Y} \in \{\mathsf{N}, \mathsf{E}\}$ and $w \in \mathcal{D}^- \setminus \{\mathsf{N\text{-}E}\}$. Then $h_{\mathsf{Y}}^{-1}(f_{\mathsf{X}}(w)) = f_{\mathsf{X}}(h_{\mathsf{Y}}^{-1}(w))$.*

Proof. A word $w \in \mathcal{D}^-$ has format $u_{\mathsf{N}}\text{-}u_{\mathsf{E}}$ for some words $u_{\mathsf{N}}, u_{\mathsf{E}} \in \mathcal{D}$. Then

$$\begin{aligned}
h_{\mathsf{Y}}^{-1}(f_{\mathsf{X}}(u_{\mathsf{N}}\text{-}u_{\mathsf{E}})) &= h_{\mathsf{Y}}^{-1}(u_{\mathsf{X}}\text{-}u_{\mathsf{N}}u_{\mathsf{E}}) \\
&\overset{L.\ 3.1}{=} h_{\mathsf{Y}}^{-1}(u_{\mathsf{X}})\text{-}h_{\mathsf{Y}}^{-1}(u_{\mathsf{N}}u_{\mathsf{E}}) \\
&\overset{L.\ 3.1}{=} h_{\mathsf{Y}}^{-1}(u_{\mathsf{X}})\text{-}h_{\mathsf{Y}}^{-1}(u_{\mathsf{N}})h_{\mathsf{Y}}^{-1}(u_{\mathsf{E}}) \\
&= f_{\mathsf{X}}(h_{\mathsf{Y}}^{-1}(u_{\mathsf{N}})\text{-}h_{\mathsf{Y}}^{-1}(u_{\mathsf{E}})) \\
&\overset{L.\ 3.1}{=} f_{\mathsf{X}}(h_{\mathsf{Y}}^{-1}(u_{\mathsf{N}}\text{-}u_{\mathsf{E}})).
\end{aligned}$$

□

Now we have all the ingredients to detect directions on the compass by means of h_{N} and h_{E}. In general, a direction w is derived from $\mathsf{N\text{-}E}$ by consecutively applying n mappings $f_{\mathsf{Y}_1}, \ldots, f_{\mathsf{Y}_n}, \rho$ for some integer $n \geq 0$ and subscripts $\mathsf{Y}_1, \ldots, \mathsf{Y}_n \in \{\mathsf{N}, \mathsf{E}\}$. In other words, $w = (\rho \circ f_{\mathsf{Y}_n} \circ \cdots \circ f_{\mathsf{Y}_1})(\mathsf{N\text{-}E})$. If we apply $h_{\mathsf{X}_1}^{-1}$ to w, for some $\mathsf{X}_1 \in \{\mathsf{N}, \mathsf{E}\}$, we can shift $h_{\mathsf{X}_1}^{-1}$ in the sequence of applied $\rho \circ f_{\mathsf{Y}_n} \circ \cdots \circ f_{\mathsf{Y}_1}$ just before f_{Y_1}, using Lemma 3.1 and Observation 3.4. Then, by Observation 3.3, $h_{\mathsf{Y}_1}^{-1}$ and f_{Y_1} cancel out if and only if $\mathsf{X}_1 = \mathsf{Y}_1$. In other words,

$$(h_{\mathsf{Y}_1}^{-1} \circ \rho \circ f_{\mathsf{Y}_n} \circ \cdots \circ f_{\mathsf{Y}_2} \circ f_{\mathsf{Y}_1})(\mathsf{N\text{-}E}) = (\rho \circ f_{\mathsf{Y}_n} \circ \cdots \circ f_{\mathsf{Y}_2})(\mathsf{N\text{-}E}). \tag{1}$$

If $\mathsf{X}_1 \neq \mathsf{Y}_1$, then the term ends up not being defined. By applying this transformation iteratively, we obtain the following.

Theorem 3.5. *Let* $\mathsf{X}_1, \ldots, \mathsf{X}_n, \mathsf{Y}_1, \ldots, \mathsf{Y}_n \in \{\mathsf{N}, \mathsf{E}\}$ *for a positive integer n. Then* $(h^{-1}_{\mathsf{X}_n} \circ \cdots \circ h^{-1}_{\mathsf{X}_2} \circ h^{-1}_{\mathsf{X}_1} \circ \rho \circ f_{\mathsf{Y}_n} \circ \cdots \circ f_{\mathsf{Y}_2} \circ f_{\mathsf{Y}_1})(\mathsf{N\text{-}E})$ *equals* NE *if* $(X_1, \ldots, \mathsf{X}_n) = (\mathsf{Y}_1, \ldots, \mathsf{Y}_n)$*, and is not defined otherwise.*

Proof. by Equation (1), $(h^{-1}_{\mathsf{X}_n} \circ \cdots \circ h^{-1}_{\mathsf{X}_2})(h^{-1}_{\mathsf{X}_1} \circ \rho \circ f_{\mathsf{Y}_n} \circ \cdots \circ f_{\mathsf{Y}_2} \circ f_{\mathsf{Y}_1})(\mathsf{N\text{-}E})$ is only defined for $\mathsf{X}_1 = \mathsf{Y}_1$, and is equal to $(h^{-1}_{\mathsf{X}_n} \circ \cdots \circ h^{-1}_{\mathsf{X}_2} \circ \rho \circ f_{\mathsf{Y}_n} \circ \cdots \circ f_{\mathsf{Y}_2})(\mathsf{N\text{-}E})$. By repeating this argument for $h^{-1}_{\mathsf{X}_2}, h^{-1}_{\mathsf{X}_3}, \ldots, h^{-1}_{\mathsf{X}_n}$, the statement follows. □

Hence, given that we can consecutively apply $h^{-1}_{\mathsf{X}_1}, h^{-1}_{\mathsf{X}_2}, \ldots, h^{-1}_{\mathsf{X}_n}$ to some word w and eventually obtain NE, certifies that $w \in \mathcal{D}$ and that $f_{\mathsf{X}_1}, \ldots f_{\mathsf{X}_n}$ are the applied rules. Vice versa, any direction $(\rho \circ f_{\mathsf{Y}_n} \circ \ldots f_{\mathsf{Y}_2} \circ f_{\mathsf{Y}_1})(\mathsf{N\text{-}E}) \in \mathcal{D}$ can be transformed to NE by consecutively applying $h^{-1}_{\mathsf{Y}_1}, h^{-1}_{\mathsf{Y}_2}, \ldots, h^{-1}_{\mathsf{Y}_n}$. By Lemma 3.2, we then have that exactly one of h^{-1}_{N} and h^{-1}_{E} is applicable to any direction $\mathcal{D} \setminus \{\mathsf{NE}\}$, and hence each direction has its unique derivation in terms of the sequence of f_{E} and f_{N}. In other words, each direction $\mathcal{D}$ defines a unique direction on the compass.

4 Algorithms

First, we observe a *simple* but slow algorithm, then we develop a *fast* algorithm. The findings of the last section immediately imply the *simple* algorithm: If the input is NE, we accept it as a direction. Otherwise, if none or both of h^{-1}_{N} and h^{-1}_{E} are applicable to the input, we may reject the input as not being a direction. Otherwise, we apply the unique mapping that is applicable to the input and recurse. Whichever mapping was applicable, reveals the derivation from the initial direction NE. For an input w of length n, this approach parses about n times a word of length at most length n, which summarizes to a run time of $O(n^2)$. This analysis is tight as shown by the example $\mathsf{N}^k\mathsf{E}$ for an integer k, where, with every recursion, the input length shrinks only by one.

Now we develop the *fast* algorithm. Sometimes the length of a word shrinks considerably when h^{-1}_{N} or h^{-1}_{E} is applied, such that we can hope for fewer than n recursions. We consider three cases.

Case Infix $\mathsf{N}^k\mathsf{E}$, $k \geq 2$ Consider the extreme case of $\mathsf{N}^k\mathsf{E}$ for some integer $k \geq 2$. Then $k - 1$ steps are needed to decompose the input to NE:

$$\mathsf{N}^k\mathsf{E} \underbrace{\overset{h^{-1}_{\mathsf{N}}}{\to} \mathsf{N}^{k-1}\mathsf{E} \overset{h^{-1}_{\mathsf{N}}}{\to} \ldots \overset{h^{-1}_{\mathsf{N}}}{\to}}_{k-1 \text{ applications of } h^{-1}_{\mathsf{N}}} \mathsf{NE}$$

The situation is similar if the input w has infix $\mathsf{N}^k\mathsf{E}$. In this case, the rules for computing h_{N}^{-1} and h_{E}^{-1} imply that w can only be decomposed by $h_{\mathsf{Y}_k}^{-1} \circ \cdots \circ h_{\mathsf{Y}_1}^{-1}$ if $(\mathsf{Y}_k, \ldots, \mathsf{Y}_1) = (\mathsf{N}, \ldots, \mathsf{N})$.

Observation 4.1. *Let $u, v \in \{\mathsf{N}, \mathsf{E}\}^*$ and integer $k \geq 2$. Then $(h_{\mathsf{Y}_{k-1}}^{-1} \circ \cdots \circ h_{\mathsf{Y}_1}^{-1})(u\mathsf{N}^k\mathsf{E}v)$ equals $(h_{\mathsf{Y}_{k-1}}^{-1} \circ \cdots \circ h_{\mathsf{Y}_1}^{-1})(u)\mathsf{NE}(h_{\mathsf{Y}_{k-1}}^{-1} \circ \cdots \circ h_{\mathsf{Y}_1}^{-1})(v)$ if $(\mathsf{Y}_{k-1}, \ldots, \mathsf{Y}_1) = (\mathsf{N}, \ldots, \mathsf{N})$, and is not defined otherwise.*

Thus, if the input w has infix $\mathsf{N}^k\mathsf{E}$ for some $k \geq 2$, then w may only consist of blocks of form $\mathsf{N}^{k'}\mathsf{E}$ where $k' \geq k-1$. Indeed, otherwise w has prefix $\mathsf{N}^\ell\mathsf{E}$ or infix $\mathsf{EN}^\ell\mathsf{E}$ for some integer $\ell < k-1$. That means that w is derived from a word w' that has prefix E or has infix EE. By induction, one can show that w' must be derived from ENE. However, any word with an infix NN, such as w, must be derived from NNE, contradicting that w is derived from w'.

These findings justify the following subroutine. Parse the input w from left to right to determine the largest integer k such that w contains infix $\mathsf{N}^k\mathsf{E}$. This can be done in time linear in n, the length of w. We know that a direction w has form $\mathsf{N}^{k_1}\mathsf{E}\ \mathsf{N}^{k_2}\mathsf{E} \ldots \mathsf{N}^{k_m}\mathsf{E}$ for some integers $k_1, k_2, \ldots, k_m \in \{k, k-1\}$ and $m \geq 1$. If that is not the case, we can safely reject the input w. Otherwise, we know that w decomposes to $(h_{\mathsf{N}}^{-1})^{k-1}(w) = u_1 \ldots u_m$ where $u_i = \mathsf{E}$ for $k_i = k-1$ and where $u_i = \mathsf{NE}$ for $k_i = k$. An example with $k = 3$ is

$$\mathsf{NNNENNENNNE} = \mathsf{N}^3\mathsf{EN}^2\mathsf{EN}^3\mathsf{E} \overset{(h_{\mathsf{N}}^{-1})^2}{\rightarrow} \mathsf{NEENE}.$$

Hence, by a second parse of w, we can determine the integers $k_1, \ldots, k_m \in \{k, k-1\}$ and then directly state $(h_{\mathsf{N}}^{-1})^{k-1}(w)$, hence perform $k-1$ decompositions steps at once. The run time is linear in n. Let us perform this step whenever $k \geq 2$. Then every part $u_1, \ldots, u_m$ is shortened to a part $v_1, \ldots, v_m$ that shrunk by a factor of at least $2/3$. A border case is NNE that is only shrunk to NE.

Case Infix $\mathsf{E}^k\mathsf{NE}$, $k \geq 2$ We have a similar situation if the input word w contains a long sequence of E's. Such a maximum length infix E^k with $k \geq 2$ must be followed by NE, as h_{E} must have been the last derivation step. Consider that w has form $u\mathsf{E}^k\mathsf{NE}v$ for some $u, v \in \{\mathsf{N}, \mathsf{E}\}^*$. Then $h_{\mathsf{E}}^{-1}(u\mathsf{E}^k\mathsf{NE}v) = h_{\mathsf{E}}^{-1}(u)\,\mathsf{N}^k\mathsf{E}\,h_{\mathsf{E}}^{-1}(v)$. Now we are in the same case as discussed before, which is that the input has an infix $\mathsf{N}^k\mathsf{E}$. Hence, our algorithm may determine the largest k such that the input has infix $\mathsf{E}^k\mathsf{NE}$. We decompose once via h_{E}^{-1} and then proceed as we did for the case infix $\mathsf{N}^k\mathsf{E}$, $k \geq 2$. Then again the input shrinks by a factor of at least $2/3$. Similarly to before, this step is possible in time $O(n)$.

Remaining Case In the remaining case, the input w neither has infix N^k nor infix E^k, for every $k \geq 2$. Then w has form $u_1 \dots u_m$ where $u_i \in \{\mathsf{NE}, \mathsf{ENE}\}$ for every $i \in \{1, \dots, m\}$. That is, w decomposes to $h_{\mathsf{E}}^{-1}(u) = v_1 \dots v_m$ where $v_i = \mathsf{E}$ if $u_i = \mathsf{NE}$, and where $v_i = \mathsf{NE}$ if $u_i = \mathsf{ENE}$; for $i \in \{1, \dots, m\}$. Similarly to before, our algorithm can perform this step in time linear in the input length. Crucially, we again shorten the word length by at least a factor of $2/3$, since every part is shortened by at least this factor. For example,

$$\mathsf{NE}\,\mathsf{NE}\,\mathsf{ENE} \overset{h_{\mathsf{E}}^{-1}}{\to} h_{\mathsf{E}}^{-1}(\mathsf{NE})h_{\mathsf{E}}^{-1}(\mathsf{NE})h_{\mathsf{E}}^{-1}(\mathsf{ENE}) = \mathsf{EENE}.$$

Run Time To summarize, three cases of the currently considered word w are possible. Either there is a long sequence of N's, a long sequence of E's, or w has a simple format. In every case, we can perform one or more decomposition steps at once in time linear in the length of the current word. The result is that recursively we shrink the input length by factor of $2/3$. Therefore, the recurrence relation for the run-time depending on the initial input length n is $T(n) = T(\frac{2}{3}n)O(n)$. This recurrence relation is solved by the third case of the master theorem, as for example presented in the text book by Cormen et al. [4]. (That is $T(n)$ has form $T(n) = a \cdot T(\frac{n}{b}) + f(n)$ with $f(n) \in O(n)$ with $a = 1$ and $b = \frac{3}{2}$. We have that $f(n) \in \Omega(n^{\log_{3/2} a+\varepsilon})$ for $\varepsilon = 0.5 > 0$ and $af(\frac{n}{b}) < c \cdot f(n)$ for $c = \frac{4}{3} < 1$ for all sufficiently large n.) Hence we conclude that the overall run time is $T(n) = O(n)$.

5 Connection to Rewriting Systems

The key step to understand the directions $\mathcal{D}$ on a compass was rephrasing them in terms of a rewriting system. As a consequence of Theorem 3.5, that is:

- $\mathsf{NE} \in \mathcal{D}$, and
- $h_{\mathsf{X}}(u) \in \mathcal{D}$ if $u \in \mathcal{D}$ and $\mathsf{X} \in \{\mathsf{N}, \mathsf{E}\}$.

Hence $\mathcal{D}$ is defined by a *context-free tabled Lindenmayer system* (0L-system) on the alphabet $\{\mathsf{N}, \mathsf{E}\}$, start-word NE and mappings $\{h_{\mathsf{N}}, h_{\mathsf{E}}\}$. Here *context-free* refers to that the production rules defined by $h_{\mathsf{N}}, h_{\mathsf{E}}$ are context-free and does not imply that $\mathcal{D}$ is context-free. Further, *tabled* refers two that there is more then one way to rewrite a word at any given step, here either by h_{N} or by h_{E}. L-systems were introduced by Lindenmayer in 1968 to describe growth of multicellular organisms [6]. We point to the monograph on this matter of Prusinkiewicz & Lindenmayer [7]. The properties and expressiveness of different kinds of L-systems were extensively studied, see for example the book chapter of Kari et al. [5]. As a tabled

0L-system, $\mathcal{D}$ is a context-sensitive language [8]. On the other hand, we can observe that $\mathcal{D}$ is not a context-free language by considering the pumping lemma [1] on the word $\mathsf{N}^k\mathsf{EN}^k\mathsf{ENN}^k\mathsf{E} \in \mathcal{D}$ for an integer k. Nevertheless, as we saw, the directions $\mathcal{D}$ can be recognized in linear time.

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