POPULAR MATCHING IN ROOMMATES SETTING IS NP-HARD *

Sushmita Gupta[†] Pranabendu Misra[‡] Saket Saurabh[§]
Meirav Zehavi[¶]

Abstract

An input to the POPULAR MATCHING problem, in the roommates setting, consists of a graph G where each vertex ranks its neighbors in strict order, known as its preference. In the POPULAR MATCHING problem the objective is to *test* whether there exists a matching $M^\star$ such that there is no matching M where more people (vertices) are happier (in terms of the preferences) with M than with $M^\star$. In this paper we settle the computational complexity of the POPULAR MATCHING problem in the roommates setting by showing that the problem is NP-complete. Thus, we resolve an open question that has been repeatedly asked over the last decade.

1 Introduction

Matching problems with preferences are ubiquitous in everyday life scenarios. They arise in applications such as the assignment of students to universities, doctors to hospitals, students to campus housing, pairing up police officers, kidney donor-recipient pairs and so on. The common theme is that individuals have preferences over the possible outcomes and the task is to find a matching of the participants that is in some sense optimal with respect to these preferences. In this

*A preliminary version of this paper appeared in the proceedings of SODA 2019 [12] and preprint is available at `arxiv.org/abs/1803.09370`. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement no. 819416 and no. 715744), and from the Israel Science Foundation (ISF) under the Individual Research Grant program (grant no. 1176/18).

†National Institute of Science Education and Research (NISER), Bhubaneshwar, India. `sushmitagupta@niser.ac.in`

‡Max Planck Institute for Informatics. Germany. `pmisra@mpi-inf.mpg.de`

§The Institute of Mathematical Sciences, HBNI, Chennai, India. `saket@imsc.res.in`

¶Ben-Gurion University, Beersheba, Israel. `meiravze@bgu.ac.il`

paper we study the computational complexity of computing one such solution concept, namely the Popular Matching problem. The input to the Popular Matching problem consists of a graph on n vertices and the preferences of the vertices represented as a ranked list of the neighbors of every vertex, said to be the *preference list* of the vertex. The goal is to find a *popular matching*–a matching that is preferred over any other matching (in terms of the preference lists) by at least half of the vertices in the graph. Situations in which a *stable matching* – a matching that does not admit a *blocking edge*, i.e. an edge whose endpoints prefer each other to their respective "situation" in the current matching – is too restrictive, popular matching finds applicability, since a stable matching is the smallest-sized popular matching. So for applications where it is desirable to have matchings of larger size than a stable matching – for instance, allocating projects to students, or pairing up police officers, where the absence of blocking edges is not mandatory – popular matching may be a suitable alternative. The notion of popularity captures a natural relaxation of the notion of stability: blocking edges are permitted but the matching, nevertheless enjoys "global stability".

We define the Popular Matching problem formally as follows. Let G denote be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $N_G(v)$ denote the neighborhood of a vertex $v \in V(G)$. Given a vertex $v \in V(G)$, a *preference list of v in G* is a *bijective* function $\ell_v : N_G(v) \to \{1, 2, \ldots, |N_G(v)|\}$. Informally, the smaller the number a vertex $v \in V(G)$ assigns to a vertex $u \in N_G(v)$, the more v prefers to be matched to u. In particular, for all $u, w \in N_G(v)$, if $\ell_v(u) < \ell_v(w)$, then v prefers u over w. A matching M in G is a subset of edges that do not share an endpoint. We say that a vertex $v \in V(G)$ is matched by a matching M if there exists a (unique) vertex $u \in V(G)$ such that $\{u, v\} \in M$, which we denote by $u = M(v)$.

In literature, the terminology related to Popular Matching is closely related to that of the Stable Marriage problem. When the input graph is (bipartite) arbitrary, the instance is said to be that of the *(stable marriage) roommates setting* of the problem. Roughly speaking, a vertex $v \in V(G)$ prefers a matching M over a matching M' if its "status" in M is better than the one in M', where being not matched is the least preferred status. Formally, the notion of preference over matchings is defined as follows. Given two matchings in G, denoted by M and M', we say that a vertex $v \in V(G)$ *prefers* M over M' if one of the following conditions is satisfied: *(i)* v in matched by M but not matched by M'; *(ii)* v is matched by both M and M', and $\ell_v(M(v)) < \ell_v(M'(v))$. We say that M' is *more popular* than M, if the number of vertices that prefer M' to M exceeds the number of vertices that prefer M to M'. A matching M is *popular* if and only if there is no matching M' that is more popular than M. In the decision version of the Popular Matching problem, given an instance $I = (G, L = \{\ell_v : v \in V(G)\})$, *the question is whether there exists a popular matching?*

History of the problem and our result. The provenance of the notion of a popular matching can be dated to the work of Condorcet in 1785 on the subject of a *Condorcet winner* [8]. In the last century, however, the notion was introduced as the *majority assignment* by Gärdenfors [11] in 1975. Abraham *et al.* [2] was the first to discuss an efficient algorithm for computing a popular matching albeit for the case where the graph is bipartite and only the vertices in one of the partitions have a preference list, a setting known as the *housing allocation*. The persuasive motivation and elegant analysis of Abraham *et al.* led to a spate of papers on popular matching [25, 15, 22, 20, 21, 3, 16, 23] covering diverse settings that include strict preferences as well as one with ties. It is well-known that when the input graph is bipartite–the stable marriage setting– Popular Matching can *always* be decided affirmatively in polynomial time, [14]. It is equally well-known that when the graph is arbitrary, the computational complexity of Popular Matching is unknown. In particular, whether Popular Matching is NP-hard has been repeatedly, explicitly posed as an open question over the last decade [1, 3, 5, 7, 13, 14, 16, 17, 18, 23, 24, 26]. Indeed, it has been stated as one of the main open problem in the area (see the aforementioned citations). In this paper we settle this question by proving the following result.

Theorem 1. Popular Matching *is NP-complete.*[1]

Our method. An optimization question related to the Popular Matching is about finding a popular matching of the largest size (as not all popular matchings are of same size). Let this problem be called Max-Sized Popular Matching. Until recently, it was also not known whether this problem is NP-hard in roommate setting. Recently, Kavitha showed Max-Sized Popular Matching in arbitrary graphs is NP-hard, [18]. This reduction serves as one of the main three gadgets in our reduction–the other two gadgets are completely new. The design of our reduction required several new insights. Firstly, our source problem is a "3-SAT-like" variant of Vertex Cover, called the Partitioned Vertex Cover, which allows us to enjoy benefits of both worlds: we gain both the lack of "optimization constraints" as in 3-SAT, and the simplicity of Vertex Cover.

The input of Partitioned Vertex Cover consists of a graph G, a collection $\mathcal{P}$ of pairwise vertex-disjoint edges in G, and a collection $\mathcal{T}$ of pairwise vertex-disjoint triangles in G such that every vertex in $V(G)$ occurs in either a triangle in $\mathcal{T}$ or an edge in $\mathcal{P}$ (but not in both). In other words, $\mathcal{T} \cup \mathcal{P}$ forms a partition of $V(G)$ into sets of sizes 3 and 2. The objective is to decide whether G has a vertex cover $U \subseteq V(G)$ such that for every $P \in \mathcal{P}$, it holds that $|U \cap P| = 1$ and for every $T \in \mathcal{T}$, it holds that $|U \cap T| = 2$.

[1] After a preprint of this paper appeared on arXiv, Kavitha [19] also published a pre-print showing this theorem. Her reduction is quite different from this paper, and it also appears in the proceedings of SODA 2019 [9].

The use of PARTITIONED VERTEX COVER as the source problem neccessitates us to encode a selection of *exactly* one "element" out of two, and exactly two "elements" out of three. Here, our gadget design is carefully tailored to exploit a known characterization of a popular matching [14, Theorem 1] in terms of admissible alternating paths and cycles in a graph associated with a candidate matching (to be a popular matching). In particular, we make use of "troublemaker triangles"–these are triangles consisting of three vertices, one of whom must be matched to a vertex outside the triangle to give rise to a popular matching. We embed these triangles in a structure that coordinates the way in which they can be *traversed* in the aforementioned characterization. Our gadgets lead traversals of such paths and cycles to dead-ends.

Related results. Chung [4] was the first to study the POPULAR MATCHING problem in the roommates setting. He observed that every stable matching is a popular matching. In the midst of a long series of articles, the issue of the computational complexity of POPULAR MATCHING in an arbitrary graph remained unsettled, leading various researchers to devise notions such as the *unpopularity factor* and *unpopularity margin* [15, 27, 13] in the hope of capturing the essence of popular matchings. A solution concept that emerged from this search is the *maximum sized popular matching*, motivated by the fact that unlike stable matchings (Rural Hospital Theorem [28]), all popular matchings in an instance do not match the same set of vertices or even have the same size. Thus, it is natural to focus on the size of a popular matching. There is a series of papers that focus on the MAX-SIZED POPULAR MATCHING problem in bipartite graphs (without ties in preference lists) [17, 14, 7] and (with ties) [6]. When preferences are strict, there are various polynomial time algorithms that solve MAX-SIZED POPULAR MATCHING in bipartite graphs: Huang and Kavitha [13] give an $O(mn_0)$ algorithm that is improved by Kavitha to $O(m)$ [17] where m and n_0 denote the number of edges in the bipartite graph and the size of the smaller vertex partition, respectively. In the presence of ties (even on one side), the MAX-SIZED POPULAR MATCHING was shown to be NP-hard [6]. It is worth noting that every stable matching is popular, but the converse is not true. As a consequence of the former, every bipartite graph has a popular matching that is computable in polynomial-time because it has a stable matching computable by the famous Gale-Shapley algorithm described by the eponymous authors in their seminal paper [10].

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