Book Introduction by the Authors

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Computation Counts – An Introduction to Analytic Concepts in Computer Science[*]

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Overview

The study of Computer Science requires some familiarity with many mathematical topics. Traditionally undergraduate programmes have focused on what are dubbed "Discrete Methods": Boolean algebra and logic; sets and relations; combinatorics; graph theory; probability, etc. One consequence has been the availability of numerous excellent and carefully presented textbooks having the common aim of providing clear expositions of Discrete Methods and taking particular care respecting the needs and background of first-year CS students.

When it comes to fields such as Calculus, Vector and Matrix algebra, Complex Number Theory, and Spectral Methods, however, a rather different picture emerges. Although in modern CS, research areas such as Optimization, Computational Game Theory, Data Science, Machine Learning, etc., often exploit mathematical machinery originating in what I shall refer to as "Analytic Concepts", there appear to be very few resources offering accessible and relevant introductions. Such texts as are available being either intended for specialist mathematics students or directed towards motivating applications in non-computing disciplines such as Chemistry or Biological Sciences.

It is increasingly the case that Analytic Concepts as described in the preceding paragraph feature as part of a first or second year undergraduate CS degree programme. This may especially hold true of institutions offering specialist modules in Data Science, advanced AI, or Optimization. The text of the present book is based on material from a module (Analytic Techniques for CS) taught by the au-

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thor to first-year students at the University of Liverpool. This was introduced in January 2018 and adds to an existing first semester Discrete Methods module.

This book is intended to address the gap between the widespread availability of Discrete Methods textbooks and the comparative lack of relevant texts with respect to Analytic Concepts. Its rationale is twofold: firstly, to present the basic concerns and techniques of a number of mathematical topics not usually dealt with on Discrete Methods courses; secondly, to do so in a manner which makes clear through examples why the specific themes are essential elements of CS.

A key theme promoted throughout the text (and emphasized in the volume's title) is that the fundamental computational question is "How do I count and measure this object?". This, of course, is a rather different perspective from, most importantly, that of Claude Shannon in [3], whereby the fundamental computational question is posed as "What is information?". I think both of these are perfectly justifiable claims and I have no intention of disputing Shannon's view. Where there is a distinction between the two lies in the interests of the audiences to whom the question is posed. The challenge raised by "What is information?" asks for a solution to an abstract issue of huge importance to the formal basis of Computer Science as a *Science*. The question "How do I count and measure this object?" deals with rather more mundane, but no less important, matters: namely with how computation is used to assist with practical issues. Thus, on one side "What is information?" concerns computation as a *science*; on the other "'How do I count and measure?" is the currency of computation as a *day-to-day activity*.

Organization

The book consists of eight chapters comprising an Introduction, six specialist technical chapters, and a concluding reprise.

The structure of each technical chapter has a similar pattern.

a. An introduction to the topic under consideration that in some cases (e.g. Chapters 2, 4, and 5) includes an overview of historical development, while in others (Chapters 3, 6 and 7) presents some contextualization with regard to CS applications.

b. The main technical presentation (including some worked examples of the relevant techniques).

c. A short summary of the main ideas discussed including pointers to where these will be developed later in the text. For example Chapter 2, among whose concerns are properties of polynomials, does not discuss the matter of the roots of $x^2 + 1$ but signals its treatment and further discussion in Chapter 5.

d. An outline of *projects* describing various software activities.

e. A section of "*Endnotes*". These provide further commentary on some points highlighted in the main body. The matters discussed are not essential to understanding (being included solely for the benefit of interested readers) and would be likely to disrupt the flow of the main text. In general, these may be ignored without losing essential supporting material.

It should be noted that reinforcement of practical experience is done through the, sometimes rather large scale, activities offered within the Projects sections. There are no lengthy screeds of variation-on-a-theme exercises such as "enumerate the coefficients of (followed by twenty or more polynomials)", or "find the first derivative and classify the critical points of (followed by another catalogue of increasingly obscure functions)". My personal belief (concerning which I can appreciate others may disagree) is that such hand-cranking repetition does little to enhance practical understanding. Actually *implementing* an algorithm to form the quotient and remainder of two polynomials described through their respective coefficients seems, to me, to provide at least as great an appreciation of the mechanics of polynomial division as would be given by laboriously working through a dozen or more specific examples.

Chapter 1 – Introduction

In this chapter a high-level overview of the specialist topics addressed in the remainder of the text is presented. In order to place these in context a brief survey of how the methods considered have been applied in a selected number of other scientific disciplines is presented. The chapter concludes by summarizing the required background knowledge assumed.

Chapter 2 – The Idea of Number

This builds on the consequences of two critical observations. Firstly that the concept of number and its use was a hugely significant cultural development on a par with the evolution of spoken and written language. Secondly that the notion of "number" in purely abstract terms is computationally powerless: some means of *representing* numerical information is also needed. Pursuing the second of these leads to a review of classical number representation methods (tally schemes, Greek and Roman systems) together with a discussion of their problematic aspects. The growth of modern styles through the watershed discovery of positional schemes including an explicit representation of zero is then discussed.

An important distinction is made between the concept of some numerical quantity (x say) and the representation of that quantity in some formalism, e.g. $\rho(x)$ where ρ could be anything from Natural language, Roman numerals or contemporary decimal schemes. As a simple illustration of computational aspects some basic algorithms for translating between x (a numerical quantity) and $(x)_{base}$ (the number x described in some finite base) are given.

The next section of this chapter moves on to describe ideas behind differing types of number (Natural, Whole, Integer, Rational and Real) discussing origins and notational conventions. This leads to a treatment of the formal structure of polynomials of a given degree and their properties. In addition to introducing ideas relating to polynomial roots a range of operations on polynomial forms (product, division, factorization etc.) are offered. The principal ideas are described using univariate forms, however, a brief summary of multivariate structures is included.

Chapter 3 – Vectors and Matrices

The premise of this chapter is that there are contexts requiring some form of ordered structure to be added to the basic notion of number type. It presents vectors (distinguishing some differences from the relational algebra concept of n-tuple), operations on vectors such as scalar and cross product together with an introduction to the notion of vector space. The concept of differing types of vector transformation is then introduced. The relation between linear transformations and matrix-vector product is described followed by a description of affine transformations. The class of projective transformations is mentioned, however, detailed description is avoided.

A core application of vector and matrix methods in CS arises in realizing basic graphics effects. After describing simple (2×2, 3×3) matrix forms these are illustrated together with homogenous coordinates by giving matrix-vector implementations of effects such as scaling, rotation, translation and shearing.

Chapter 4 – Basic introduction to Calculus

This introduces elements of Differential Calculus emphasizing its importance in Optimization Theory. After motivating the idea of first derivative in geometric terms, the second derivative test is described and applied to some examples.

In real environments it is rarely the case that Optimization involves finding a setting for just a single variable. In order to give some flavour of how multivariable optimization could be handled, the concept of partial differentiation is introduced with an overview of how optimization problems can be tackled through the analysis of partial derivatives. Some example cases are then worked through and potential issues with extension to large numbers of variables discussed.

The concept of roots of a polynomial had been described in Chapter 2 and the importance of Differential Calculus to root finding techniques is considered. This is via comparison of two widely-used methods: Halley's which subject to some technical conditions can be used to find roots (zeros) of arbitrary functions and Laguerre's which is specialized to discovering roots of polynomials. The Jenkins-Traub polynomial root finding method (from [1, 2]) while referred to is not discussed in depth.

The chapter concludes with a short overview of Integral Calculus concentrating on its application as a measurement technique. The connection between Integral and Differential Calculus is presented through the concept of antiderivative so that (as a sophisticated extension of the so-called "Method of Exhaustion") the area spanned by a function between two values may be developed using the antiderivative. Discussion of the more creative techniques applied to find antiderivatives is avoided, however, the extension of basic area measurement to line lengths and volumes is briefly considered.

Chapter 5 – Complex Numbers

Here an introduction to Complex Numbers is offered. The starting point concerns the (apparent) lacuna in the review of polynomial properties from Chapter 2: since polynomials of degree k have k roots (allowing for multiplicity) what are the two roots of $x^2 + 1$?

I am sure I am not alone in noticing the level of suspicion that is sometimes encountered among students (especially those with minimal previous mathematical background) when these are just baldly stated to be $\{\imath, -\imath\}$. Since this level of scepticism was felt by figures as distinguished as René Descartes whose comments gave rise to the term "*Imaginary*" number, some detailed discussion of the basis for this suspicion and an attempt at rebuttal are presented.

The chapter continues by giving the standard alternative representations of Complex Numbers (including Argand diagrams, polar coordinates and Euler form) together with basic operations on these.

One of the problematic aspects of Complex analysis is the care needed when dealing with Complex powers (z^w). A standard example of fallacious treatment is given: evaluating $(e^{2\pi\imath})^\imath$ so that both 1 and $e^{-2\pi}$ appear to be justifiable. A full breakdown of cases is not given: instances of raising a Complex number to a Complex power being uncommon in CS. The applicable analyses where at least one element of z^w is Real are presented. Discussion of Complex powers leads to the important idea of primitive roots of unity.

Computational applications arising from concepts grounded in Complex numbers are seen to be of considerable importance. In order to emphasize this fact the chapter includes a variety of such applications. Among other topics this presents: an overview of quaternion algebra and its role in sophisticated 3-dimensional graphics rendering; a detailed discussion of the Fourier Transform (in its discrete form) together with its application to image analysis and integer multiplication. One final application is found in iterating Complex functions in such a way that these lead to intricate graphical objects such as Mandelbrot and Julia-Fatou sets. Perhaps less well-known is the use of Julia-Fatou sets in the AI field of Algorithmic Composition, e.g. Walker [4].

Chapter 6 – Statistics and Data Analysis

This focuses on Statistics and methods in Data Analysis (regression and correlation coefficients). Its aim is to highlight the importance of experimental method in CS. After reviewing the distinctive aims of Probability Theory and Statistics, some foundational ideas (population, random variable, expectation etc.) are given. This chapter then looks at the Normal Distribution and some selected discrete distributions (Binomial, Geometric, Poisson). The summary of discrete distributions is followed by a review of how moments of a distribution have been used in CS. Here Markov's inequality is presented together with one of the standard first moment arguments applied in analyzing 3-colourability. After discussing some common statistical fallacies the concept of Statistical Confidence is presented together with the elements of some tools used to analyze confidence (Student t-test, Welch's Test) and a small number of example studies. The chapter concludes with an overview of regression methods and other data analysis techniques. This includes a presentation of linear least squares regression, its extension to a small number of non-linear functions (quadratic regression being dealt with separately), polynomial interpolation via Lagrange interpolants, and measures such as Spearman's and Pearson's correlation coefficients.

Chapter 7 – Introduction to Spectral Methods

Chapter 7 returns to the topic of Matrix Theory with a particular emphasis on spectral methods. It describes various computational views of matrix determinant, introduces the eigenvalue problem and the Perron-Frobenius Theorem. A number of computational methods for approximating eigenvalues and eigenvectors are discussed: Power and Inverse Power methods as well as Hotelling's Deflation. Jacobi's transformation of symmetric matrices to triangular is described at a high-level only.

The principal focus, however, is towards important applications of spectral analysis in CS. Among those presented are Google's page rank technique using not only the basic ranking method under the assumption that no pages form "dead-ends" but also the modifications adopted in order to ensure the presence of a dominant eigenvalue.

Further applications are offered through the matrix description form known as Singular Value Decomposition (svd). The use of svd as a method of (lossy) Data Compression is illustrated by showing its effect when applied to a series of images.

As another application, included to emphasize that spectral methods continue to find new uses in CS, some recent work from Computational Argument using spectral techniques to model unruly debate is described.

Chapter 8 – Epilogue

This concluding part reviews the material presented over the previous chapters with the aim of placing the different topics in the context of developments in CS. Thus the treatment of polynomials is considered with respect to influence on Computer Algebra and Automated Theorem proving; the application of Complex Numbers to novel ideas in AI (such as Algorithmic Composition). This chapter also discusses two significant areas which have not featured in the main body: Numerical Methods and Information Theory and the rationale for their omission.

Summary

The aim of this book is to provide an introduction to mathematical ideas which are of significance in more advanced CS studies. It does not assume mathematical background beyond what would have been covered at school or within an introductory Discrete Methods course. In order to promote accessibility, technical and detailed mathematical exposition is kept to a minimum. The core rationale is to show why specific mathematical disciplines matter and to provide support for such claims through illustrative applications in modern CS.

Caveat

It is only fair to point out (as will, undoubtedly, be confirmed by many students who have had to contend with it) that the author has what may best be described as some “eclectic eccentricities” and an even more exaggerated, eccentric sense of humour. Some readers may consider that one or even both of these have been rather too freely indulged in the text. This would be unfortunate and I can only refer those of such an opinion to the comment on p. 460 (or to that terse rebuttal of “you-ought-not-to-have” critics embodied within John 19:22). That said, most of the more heinous examples are confined to discussion in endnotes (e.g. the “achievements” of Paulo Valmes in Chapter 2; the discussion of lateralism in Chapter 3). As I have indicated earlier, these are included for the benefit of interested readers and are not essential to understanding of the main text.

There are, however, a few worked examples (e.g. one of the multivariable optimization examples, that involving the Geometric Distribution and that concerning confidence estimates in Poisson distributions with small sample sets) where the temptation to describe the setting used proved impossible to resist.

About the author

Paul Dunne is a Professor of CS at the University of Liverpool where he has worked since 1985. He studied CS at the University of Edinburgh (1977–1981) and completed his PhD research in the area of Boolean function complexity at Warwick University (1981–1984) under Mike Paterson’s supervision. In his time at Liverpool he has had experience in teaching all levels of undergraduate from first year through to Honours year presenting courses on Computability and Complexity Theory, Algorithms, Operating Systems, and the topic of the present book. He has published research in several fields including Boolean Function complexity, phase transition phenomena, AI and Law, complexity in multiagent systems, and models of Computational Argument.

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