Phase Transition of the 2-Choices Dynamics on Core-Periphery Networks*

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Introduction

Opinion dynamics, for short *dynamics*, are simplistic mathematical models for the competition of agents' opinions on social networks [15]. In a nutshell, given a network where each agent initially supports an opinion chosen from a finite set, a dynamics is a simple rule which agents apply to update their opinion based on that of their neighbors. Proving theoretical results on dynamics is a challenging mathematical endeavor which may require the development of new analytical techniques [5, 2].

As dynamics are aimed at modeling the spread of opinions, a central issue is to understand under which conditions the network reaches a *consensus*, i.e., a state where the whole network is taken over by a single opinion [16]. In this respect, most efforts have been directed toward obtaining *topology-independent* results, which disregard the initial opinions' placement on the network [3, 4, 10].

The *trivial* example of dynamics is the so-called Voter model, in which in each round each agent copies the opinion of a random neighbor. This classical model arises in physics and computer science and, despite its apparent simplicity, some properties have been proven only recently [14, 12, 6]. Voter reaches a

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proportionate consensus, i.e., the probability of a consensus on one opinion is proportional to the initial volume of that opinion, but the dynamics is slow ($\Omega(n)$ rounds) even on networks with small diameter such as expanders and complete graphs.

The simplest non-trivial example is then arguably the 2-Choices dynamics, where at each discrete-time step each node samples two random neighbors with replacement and, if they have the same state, the node adopts that state [8]. Still, the analysis of the 2-Choices dynamics has been limited to networks with good expansion properties, and the theoretical guarantees provided so far are essentially independent from the positioning of initial opinions [9].

In this work, we aim at contributing to the general understanding of the evolution of simple opinion dynamics in richer classes of network topologies by studying the behavior of the 2-Choices dynamics both theoretically and empirically on core-periphery networks. Core-periphery networks are typical economic and social networks which exhibit a core-periphery structure, a well-known concept in the analysis of social networks in sociology and economics [7, 11], which defines a bipartition of the agents into *core* and *periphery*, such that certain key features are identified.

We consider an axiomatic framework that has been introduced in previous work in computer science [1], which is based on two parameters only, *dominance* and *robustness*. The ranges for these parameters in the theorems we obtain include the values of the parameters in the the experimental part of this work, in which our results are validated on important datasets of real-world networks.

Intuitively, the core is a set of agents that *dominates* the rest of the network by maintaining an amount of external influence on the periphery which is higher than or at least comparable to the internal influence that the periphery has on itself. Similarly, to maintain its *robustness*, namely to hold its position and stick to its opinions, the core must be able to resist the "outside" pressure in the form of external influence. To achieve that, the core must maintain a higher influence on itself than the external influence exerted on it by the periphery. Both, high dominance and high robustness, are essential for the core to be able to maintain its dominating status in the network. Moreover, it seems natural for the core size to be as small as possible. In social-network terms this is motivated by the idea that if membership in a social elite entails benefits, then keeping the elite as small as possible increases the share for each of its members.

We consider the 2-Choices in core-periphery networks when starting from *natural* initial configurations in which the core and the periphery have different opinions. Our experiments on real-world networks show that the execution of the 2-Choices tends to fall mainly within two opposite kinds of possible behavior:

- *Almost-consensus:* The opinion of the core spreads in the periphery and takes over the network in a short time, although a small number of nodes in the periphery might retain their initial opinion.
- *Metastability:* The periphery *resists* and, although the opinion of the core may continuously "infect" agents in the periphery, most of them keep the initial opinion.

By comprehending the underlying principles which govern the aforementioned phenomena, we aim at a twofold contribution:

- We seek for the first results on basic non-trivial opinion dynamics, such as the 2-Choices, in order to characterize its behavior (i) on new classes of topologies other than networks with strong expansion and (ii) as a function of the process' initial configuration.
- We look for a *dynamic* explanation for the axioms of core-periphery networks: By investigating the interplay between the core-periphery axioms and the evolution of simple opinion dynamics, we want to get insights on *dynamical properties* which are implicitly responsible for causing social and economic agents to form networks with a core-periphery structure.

Original contribution

To understand what network key properties are responsible for the aforementioned dichotomy between a long *metastable* and a fast *consensus* behavior, we theoretically investigate a class of networks belonging to the core-periphery model.

To further simplify the theoretical analysis, in our main theorem we initially consider the setting in which *agents in the core are stubborn*, i.e., they don't change their initial opinion.[1] As a corollary of the main theorem, we show how to substitute this assumption with milder hypotheses on the core's structure, i.e., assuming a certain robustness of the core to the periphery's influence.

The common difficulty in analyzing opinion dynamics is the lack of general tools which allow to rigorously handle their intrinsic nonlinearity and stochastic dependencies [15, 8, 4, 10, 16]. Hence, the difficulty usually resides in identifying some crucial key quantities for which ad-hoc analytical bounds on the expected evolution are derived. Our approach is yet another instance of such efforts: We first provide a careful bound on the expected change of the number of agents supporting

[1]We remark that the evolution of the 2-Choices dynamics, together with the latter assumption on the stubbornness of the core, can be regarded as a SIS-like epidemic model [13, 17]. In such a model, the network is the subgraph induced by the periphery and the infection probability is given by the 2-Choices dynamics, which also determines a certain probability of *spontaneous infection* (that in the original process corresponds to the fact that agents in the periphery interact with agents in the core). This interpretation of our results may be of independent interest.

a given opinion; then, together with the use of Chernoff bounds, we obtain a concentration of probability around the expected evolution. Rather surprisingly, our analysis on the concentrated probabilistic behavior turns out to identify a *phase transition* phenomenon:

> There exists a universal constant $c^\star = \frac{\sqrt{2}-1}{2}$ such that, on any core-periphery network of n agents, if the dominance parameter c_d is greater than $c^\star$, an *almost-consensus* is reached within $O(\log n)$ rounds, with high probability;[2] otherwise, if c_d is less than $c^\star$, a *metastable* phase in which the periphery retains its opinion takes place, i.e., even if the opinion of the core may continuously "infect" agents in the periphery, most of them will retain their initial opinion for $n^{\omega(1)}$ rounds with high probability.

We validate our theoretical predictions by extensive experiments on real-world networks chosen from a variety of domains including social networks, communication networks, road networks, and the web. The experiments showed some weaknesses of the core extraction heuristic used in [1]. To avoid those drawbacks, we designed a new core extraction heuristic which repeatedly calculates densest-subgraph approximations. Our experimental results on real-world networks show a strong correlation with the theoretical predictions made by our model. They further suggest an empirical threshold larger than $c^\star$ for which the aforementioned correlation is even stronger. The discrepancy between the two thresholds should be closed in future work by providing a more fine-grained theoretical analysis which does not assume the adversary's color to be monochromatic.

We remark that our investigation represents an original contribution with respect to the main line of research on *consensus*, as it shows a drastic change in behavior for the 2-Choices dynamics on an arguably *typical* broad family of social networks which is not directly characterized by expansion properties. In particular, the convergence to the core's opinion in our theoretical and experimental results is a highly nontrivial fact when compared to previous analytical works [12, 8].

Overall, our theoretical and experimental results highlight new potential relations between the typical core-periphery structure in social and economic networks and the behavior of simple opinion dynamics – both, in terms of getting insights into the driving forces that may determine certain structures to appear frequently in real-world networks, as well as in terms of the possibility to provide analytical predictions on the outcome of simplistic models of interaction in networks of agents.

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[2]We further emphasize that our analysis is not only *mean-field*. In addition to describing how the process evolves in expectation, we show that the process does not deviate significantly from how it behaves in expectation *with high probability*, i.e., with probability at least $1 - O(n^{-c})$ for some constant $c > 0$.

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