The Algorithmics Column

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How tough is toughness?

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Abstract

We survey results and open problems related to the toughness of graphs.

1 Introduction

The concept of toughness was introduced by Chvátal [34] more than forty years ago. Toughness resembles vertex connectivity, but is different in the sense that it takes into account what the effect of deleting a vertex cut is on the number of resulting components. As we will see, this difference has major consequences in terms of computational complexity and on the implications with respect to cycle structure, in particular the existence of Hamilton cycles and k-factors.

1.1 Preliminaries

We start with a number of crucial definitions and observations. Throughout this paper we only consider simple undirected graphs. Let $G = (V, E)$ be such a graph. Every subset $S \subseteq V$ induces a subgraph of G, denoted by $G[S]$, consisting of S and all edges of G between pairs of vertices of S. We use $\omega(G)$ to denote the number of *components* of G, i.e., the set of maximal (with respect to vertex set inclusion) connected induced subgraphs of G. A *vertex cut* of G is a set $S \subset V$ with $\omega(G - S) > 1$. Clearly, complete graphs (in which every pair of vertices is joined by an edge) do not admit vertex cuts, but non-complete graphs have at least one vertex cut (if u and v are nonadjacent vertices in G, then the set $S = V \setminus \{u, v\}$ is a vertex cut such that $\omega(G-S) = 2$). As usual, the (vertex) *connectivity* of G, denoted by $\kappa(G)$, is the cardinality of a smallest vertex cut of G (if G is non-complete; for the complete graph K_n on n vertices it is usually set at $n - 1$). Adopting the terminology of Chvátal [34], we say that G is *t-tough* if $|S| \geq t \cdot \omega(G - S)$ for every vertex cut S of G. The *toughness* of G, denoted $\tau(G)$, is the maximum value of t for which G is t-tough (taking $\tau(K_n) = \infty$ for the complete graph K_n on n vertices). Hence if G is non-complete, $\tau(G) = \min\{|S|/\omega(G - S)\}$,

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where the minimum is taken over all vertex cuts of G. In [79], Plummer defined a vertex cut $S \subset V$ to be a *tough set* if $\tau(G) = |S|/\omega(G-S)$, i.e., a vertex cut $S \subset V$ for which this minimum is achieved. A graph G is *hamiltonian* if G contains a *Hamilton cycle*, i.e., a cycle containing every vertex of G.

Other crucial terminology will be given later, while we refer to [22] for general undefined terms in graph theory and to [53] for general terminology in complexity. More background for the concepts and results in this paper can be found in [6, 7, 17, 18, 19].

A basic and crucial observation originating in [34] that has led to most of the research on toughness, is the simple fact that a hamiltonian graph is necessarily 1-tough. In fact, a cycle itself is already 1-tough, because deleting any set of x vertices (and the edges incident with these vertices), the remaining parts consist of at most x cycle segments contained in at most x components. On the other hand, it is easy to come up with examples of 1-tough graphs that are not hamiltonian, but it is much harder to find nonhamiltonian graphs with a higher toughness, and it seems unlikely that nonhamiltonian graphs with an arbitrarily high toughness exist.

Historically, most of the research on toughness was based on a number of conjectures in [34]. The most challenging of these conjectures is still open: Is there a finite constant t_0 such that every t_0-tough graph is hamiltonian? For a long time it was believed that this conjecture should hold for $t_0 = 2$. This '2-tough conjecture' would then imply a number of related results and conjectures. We showed in [8] that the 2-tough conjecture is false. On the other hand, we know that the more general t_0-tough conjecture is true for a number of graph classes, including planar graphs, claw-free graphs, and chordal graphs. We come back to this later. The early research in this area concentrated on sufficient degree conditions which, combined with a certain level of toughness, would yield the existence of long cycles. The survey [7] contains a wealth of results in this direction. We will not repeat too many of these results here because we thought the readership of this bulletin is more interested in the algorithmic issues.

Research on toughness has also focused on computational complexity issues. From an algorithmic point of view, it is somewhat unfortunate that the problem of recognizing t-tough graphs is coNP-complete for every fixed positive rational t ([9]). This implies that it is NP-hard to compute the toughness of a graph. On the other hand, for some important graph classes the toughness can be computed efficiently. We will come back to this later.

1.2 Some easy observations

We start with some easy facts, and with several natural questions, some of which are easy to answer. The latter are sometimes stated as exercises to stimulate the reader to think of the solutions first before we are going to present them, and to get some feeling or intuition for the concepts involved. The following should be an easy exercise, and the solution is a folklore result.

Exercise 1. *Give an expression for the toughness of a (non-complete) tree.*

Solution. Let T be a tree with maximum degree k. Then, obviously $\tau(T) \leq \frac{1}{k}$, since deleting a vertex with degree k from T yields a forest with exactly k components. It is not difficult to convince oneself that deleting any other set S of vertices from T will not yield more than $k \cdot |S|$ components: deleting the first vertex, say $v \in S$ from T, one obtains $d_T(v) \leq k$ components in $T - v$; deleting a second vertex, say $w \in S \setminus \{v\}$ (if any) from a component H of $T - v$, one obtains at most $k - 1$ additional components, since H is replaced by at most k components of $H - w$ in $T - \{v, w\}$. Repeating the argument in case $|S| \geq 3$, we conclude that, in fact the only tough sets of T consist of a single vertex of maximum degree. □

The consequence of this solution is that for trees it is easy to determine the toughness. Many natural questions might pop up now, like the following. How easy is it to determine the toughness for general graphs? Or for graphs belonging to other restricted graph classes? Is there a relation between tough sets and minimum vertex cuts?

Let us start with the last question. It is clear that for a non-complete graph G, the removal of the vertices of a minimum vertex cut yields at least two components. Hence, we obtain the following upper bound on the toughness of G: $\tau(G) \leq \frac{\kappa(G)}{2}$. This bound is usually not sharp, as can be seen already from the above result on trees. Perhaps surprisingly, the bound is sharp for claw-free graphs, as we will show as a solution to the next exercise. We first recall some definitions.

If a graph H is isomorphic to a subgraph induced by a subset $S \subseteq V$ of the graph $G = (V, E)$, then we say that H is an *induced subgraph* of G. The graph G is called *H-free* if this is not the case, so if G does not contain a copy of H as an induced subgraph. In case H is isomorphic to $K_{1,3}$, we use the more common term *claw-free* instead of H-free. Claw-free graphs are a very well-studied graph class, including the class of *line graphs*.

Exercise 2. *Show that $\tau(G) = \frac{\kappa(G)}{2}$ for a non-complete claw-free graph G.*

Solution sketch. Let S be a tough set of G with $|S| = k$. By the connectivity, every component of $G - S$ has at least $\kappa(G)$ different neighbors in S. On the other hand, since G is claw-free, every vertex of S has neighbors in at most two components of $G - S$. Hence, $\kappa(G) \cdot \omega(G - S) \leq 2k$. Thus, $\tau(G) = \frac{k}{\omega(G-S)} \geq \frac{\kappa(G)}{2}$. The statement now follows from the observation we made earlier. □

As a consequence of the above solution and well-known complexity results on connectivity, the toughness of a claw-free graph is easy to determine.

Let us return to the question whether tough sets and minimum vertex cuts are somehow related, e.g., is it the case that for every non-complete graph G there exists a tough set of G that contains a minimum vertex cut of G? We put it as another exercise to show that this is not the case.

Question 3. *Is there a tough set that contains a minimum vertex cut in every non-complete graph?*

Answer. It is easy to come up with counterexamples. Take, for instance, two vertex-disjoint copies of a complete bipartite graph $K_{m,n}$ with $m \geq 2n \geq 4$, add one new vertex v and join v by edges to the n vertices of the smallest bipartition classes of each of the two copies. It is easy to check that $\{v\}$ is the only minimum vertex cut, whereas the two sets of n vertices of the smallest bipartition class of each of the copies are the only two tough sets. □

Since each component of $G-S$ attributes at least one vertex to an *independent set* of G, i.e., a set of mutually nonadjacent vertices of G, another natural question is whether there is a relation between $\alpha(G)$, i.e., the cardinality of a largest independent set of G, and $\tau(G)$.

Exercise 4. *Show that $\tau(G) \leq \frac{n-\alpha(G)}{\alpha(G)}$ for a non-complete graph G on n vertices.*

Solution. Let $S = V \setminus I$ for a maximum independent set I of $G = (V, E)$. Then $\omega(G - S) = |I| = \alpha(G)$. Hence, $\tau(G) \leq \frac{|S|}{\omega(G-S)} = \frac{n-\alpha(G)}{\alpha(G)}$. □

One might be tempted to think that some maximum independent set of a non-complete graph G is always contained in $G - S$ for some tough set S of G. The examples in the answer to Question 3 show that this is not the case, though.

Although a tough set S of a graph G need not contain a minimum vertex cut of G, and a maximum independent set of G is not always contained in the components of $G - S$, there is a bound on $\tau(G)$ that involves both $\alpha(G)$ and $\kappa(G)$, as stated in the final exercise in this subsection.

Exercise 5. *Show that $\tau(G) \geq \frac{\kappa(G)}{\alpha(G)}$ for a non-complete graph G.*

Solution. Let S be a tough set of G. Then S is a vertex cut of G, hence $|S| \geq \kappa(G)$. Clearly, $\alpha(G) \geq \omega(G - S)$. Hence, $\tau(G) = \frac{|S|}{\omega(G-S)} \geq \frac{\kappa(G)}{\alpha(G)}$. □

In the above, we have already seen that it is computationally easy to determine the toughness of trees and claw-free graphs. In the next section we will show that it is difficult to determine the toughness of general graphs, and we will list some known results and open problems on the computational complexity of this problem restricted to graphs from specific graph classes.

2 Complexity: known results and open problems

To start this section, we will first sketch a proof that implies that determining the toughness of general graphs is an NP-hard problem. The proof is based on similar observations as in Section 1, relating independent sets in a graph G to components of $G - S$ for a tough set S of G.

2.1 NP-hardness for general graphs

The problem of determining the complexity of recognizing t-tough graphs was first raised by Chvátal [33] and later appeared in [83] and [[35], p. 429]. Consider the following decision problem, where t is any positive rational number.

> ***t*-TOUGH**
> INSTANCE : Graph G.
> QUESTION : Is $\tau(G) \geq t$?

Theorem 6 ([9])**.** *For any positive rational number t, t-TOUGH is* NP*-hard.*

Sketch of proof. The proof we sketch here is based on [9] and the remarks in [17]. In [9], first the NP-hard variant of INDEPENDENT SET (IS) [[53], p. 194] called INDEPENDENT MAJORITY was used to show that 1-TOUGH is NP-hard (or, more precisely that NOT-1-TOUGH is NP-complete). Next the latter problem was reduced to (NOT-)t-TOUGH, for any fixed positive rational number t. As noted in [17], it is easy to use similar arguments to reduce IS itself to NOT-1-TOUGH. We outline the reduction here.

Let $G = (V, E)$ be an instance of IS, with $V = \{v_1, v_2, \ldots, v_n\}$. Construct a graph G' in the following way. First add n new vertices $w_1, w_2, \ldots, w_n$, and join v_i to w_i by an edge for $i = 1, 2, \ldots, n$. Add a complete graph H on $k - 1$ new vertices and join each vertex of H to each of the vertices v_i and w_i for all $i = 1, 2, \ldots, n$. The claim is that $\alpha(G) \geq k$ if and only if $\tau(G') < 1$. This is not difficult to prove.

The next step is to reduce NOT-1-TOUGH to NOT-t-TOUGH, by the approach given in [9], that we sketch now.

Let $G = (V, E)$ be an instance of NOT-1-TOUGH, with $V = \{v_1, v_2, \ldots, v_n\}$, and let $t = \frac{a}{b}$ for integers $a, b \geq 2$. Take a vertex-disjoint isomorphic copies of G, and take n vertex-disjoint copies of a graph H consisting of $b - 1$ copies of K_r for $r \geq a(a-1)b(b-1)$. Join the i^{th} vertex of each of the copies of G (the vertices corresponding to v_i of G) to all vertices of the i^{th} copy of H by edges. Let the new graph be G'. The claim is that $\tau(G) < 1$ if and only if $\tau(G') < t$. We omit the proof.

□

It is natural to inquire whether the problem of recognizing t-tough graphs remains NP-hard for various subclasses of graphs. We will first review some known cases for which this problem is polynomially solvable.

2.2 Polynomial cases for special graph classes

As we have seen in Section1, for a claw-free graph G, $\tau(G) = \frac{\kappa(G)}{2}$ (as first shown in [75]). Hence the toughness of claw-free graphs, and consequently of line graphs, can be determined in polynomial time. Thus, while it is NP-complete to determine whether a line graph is hamiltonian [21], it is polynomial to determine whether a line graph is 1-tough. Another class of graphs for which this is the case is the class of *split graphs*, i.e., graphs for which the vertex set can be partitioned into an independent set and a *clique*, i.e., a set of mutually adjacent vertices. Determining whether a split graph is hamiltonian was shown to be NP-complete in [78]. On the other hand, the following was shown in [71].

Theorem 7. *The class of 1-tough split graphs can be recognized in polynomial time.*

Using the fact from [37, 56] that submodular functions can be minimized in polynomial time, in [86] the following was shown.

Theorem 8. *For any rational number $t \geq 0$, the class of t-tough split graphs can be recognized in polynomial time.*

In [25], we extended the above result on split graphs by showing that the toughness of $2K_2$*-free graphs* can be computed in polynomial time. These are graphs that do not contain an induced copy of $2K_2$, the graph on four vertices consisting of two vertex-disjoint edges. It is easy to see that every split graph is a $2K_2$-free graph. A graph is *chordal* if every cycle on at least four vertices contains a *chord*, i.e., an edge joining two vertices that are not adjacent on the cycle. One can also easily check that every *co-chordal graph*, i.e., every graph that is the complement of a chordal graph, is $2K_2$-free.

Theorem 9. *The toughness of a $2K_2$-free graph can be determined in polynomial time.*

Our algorithmic proof of this result is based on one easy observation, a result of Farber [48], and the (implicit) algorithm in the proof of Woeginger [86] for split graphs.

Sketch of proof. The observation we use is that a graph $G = (V, E)$ is $2K_2$-free if and only if for every $A \subset V$ at most one component of the graph $G - A$ contains edges. The result of Farber [48] states that a $2K_2$-free graph on n vertices contains at most n^2 maximal independent sets, and that all of them can be listed in time $O(n^2)$. Given a $2K_2$-free graph $G = (V, E)$ on n vertices as input, we use the following algorithm.

1. List all maximal independent sets of G using the (implicit) polynomial-time algorithm from [48]. Denote them by $I_1, I_2, \ldots, I_k$, where $k \leq n^2$.

2. For every $i \in \{1, 2, \ldots, k\}$ consider the split graph G_i obtained from G by adding all necessary edges that turn $V \setminus I_i$ into a clique in G_i. Determine the toughness $\tau_i = \tau(G_i)$ using the (implicit) polynomial-time algorithm from [86].

3. Output $t = \min\{\tau_i \mid i \in \{1, 2, \ldots, k\}\}$.

Clearly, the algorithm outputs t in polynomial time. It remains to show that $t = \tau(G)$. For this, let S be a tough set of G. Then by our observation, at most one component of $G - S$ contains edges; the others induce a nonempty independent set I in G. Clearly, $N_G(I)$ (the set of neighbors in G of the vertices of I) is also a vertex cut, with $N_G(I) \subseteq S$ and $\omega(G - N_G(I)) \geq \omega(G - S)$. If $N_G(I) \neq S$, then $|N_G(I)| < |S|$, and hence

$$\frac{|N_G(I)|}{\omega(G - N_G(I))} < \frac{|S|}{\omega(G - S)},$$

contradicting that S is a tough set of G. Thus $N_G(I) = S$. Let I_j be a maximal independent set of G containing I. Then $S \cap I_j = \emptyset$, since otherwise I_j is not independent. Let G_j be obtained from G by turning $V \setminus I_j$ into a clique, and let $\tau_j = \tau(G_j)$. Then $\omega(G_j - S) = \omega(G - S)$, and so

$$t = \min_i \tau_i \leq \tau_j \leq \frac{|S|}{\omega(G_j - S)} = \frac{|S|}{\omega(G - S)} = \tau(G).$$

For inequality in the other direction, suppose $t = \tau_r = \min_i \tau_i$, and suppose S_r is a tough set of G_r. Then $\omega(G - S_r) \geq \omega(G_r - S_r)$, since adding edges to G cannot increase the number of components of $G - S_r$. Hence

$$\tau(G) \leq \frac{|S_r|}{\omega(G - S_r)} \leq \frac{|S_r|}{\omega(G_r - S_r)} = \tau_r = t.$$

We conclude that $t = \tau(G)$, proving Theorem 9. □

While many other problems that are NP-hard in general can be solved in polynomial time on $2K_2$-free graphs, the problem of deciding whether a $2K_2$-free graph is hamiltonian is NP-complete; indeed, as we noted before, the latter problem is even NP-complete on split graphs ([78]). We refer the interested reader to [74] for more details and references to other work on $2K_2$-free graphs.

2.3 NP-hardness for special graph classes

For many subclasses of graphs, it is NP-hard to recognize t-tough graphs. For example, in [14], using similar constructions as used in the above NP-hardness proof, it was shown that it is NP-hard to recognize t-tough graphs even within the class of graphs having minimum degree almost high enough to ensure that the graph is t-tough, in the following sense.

Theorem 10. *Let $t \geq 1$ be a rational number. If $\delta \geq (\frac{t}{t+1})n$, then G is t-tough. On the other hand, for any fixed $\epsilon > 0$, it is* NP*-hard to determine whether G is t-tough for graphs G with $\delta \geq (\frac{t}{t+1} - \epsilon)n$.*

Häggkvist [57] has shown that if $\delta \geq n/2 - 2$, there is a polynomial time algorithm to determine whether G is hamiltonian. As a consequence of a result of Jung [61], a graph G on $n \geq 11$ vertices satisfying $\delta \geq n/2 - 2$ is hamiltonian if and only if G is 1-tough. It follows that 1-tough graphs can be recognized in polynomial time when $\delta \geq n/2 - 2$.

Another interesting class of graphs is the class of *bipartite graphs*. Obviously $\tau(G) \leq 1$ for any bipartite graph G. The complexity of recognizing 1-tough bipartite graphs had been raised a number of times; see, e.g., [[24], p. 119]. In [71], Kratsch et al. were able to reduce 1-TOUGH for general graphs to 1-TOUGH for bipartite graphs by using the classical Nash-Williams construction [76].

Theorem 11. *1-TOUGH remains* NP*-hard for bipartite graphs.*

Consequently, 1-TOUGH is also NP-hard for the larger class of *triangle-free graphs*, i.e., K_3-free graphs.

2.4 Toughness for regular graphs: results and open problems

An important class of graphs that has received considerable attention is the class of *regular graphs*. Note that the maximum possible toughness of an r-regular graph G is $\frac{r}{2}$, since $\tau(G) \leq \frac{\kappa(G)}{2} \leq \frac{r}{2}$.

Chvátal [34] asked for which values of r and $n > r+1$ there exists an r-regular, $\frac{r}{2}$-tough graph on n vertices, and observed that this is always the case for r even. He also conjectured that for r odd and n sufficiently large, it would be necessary that $n \equiv 0 \bmod r$, and verified this for $r = 3$. But for all odd $r \geq 5$, Doty [41] and Jackson and Katerinis [59] independently constructed an infinite family of r-regular, $\frac{r}{2}$-tough graphs on n vertices with $n \not\equiv 0 \bmod r$.

Jackson and Katerinis [59] gave a characterization of *cubic*, i.e., 3-regular, $\frac{3}{2}$-tough graphs which allowed such graphs to be recognized in polynomial time. Their characterization of these graphs uses the concept of *inflation*, introduced by Chvátal in his original toughness paper [34].

Theorem 12. *Let G be a cubic graph. Then G is $\frac{3}{2}$-tough if and only if $G = K_4$, $G = K_2 \times K_3$, or G is the inflation of a 3-connected cubic graph.*

In [55], an analogous characterization of r-regular, $\frac{r}{2}$-tough graphs for all $r \geq 1$ is conjectured, which would allow such graphs to be recognized in polynomial time.

In the opposite direction, it was established in [10] that it is NP-hard to recognize 1-tough cubic graphs. This was generalized in [11] as follows.

Theorem 13. *For any integer $t \geq 1$ and any fixed $r \geq 3t$, it is* NP*-hard to recognize r-regular, t-tough graphs.*

The complexity of recognizing r-regular, t-tough graphs remains completely open when $2t < r < 3t$, and the complexity when $r = 2t + 1$ seems especially intriguing. The following is conjectured in [11], where the authors also sketch a possible approach to proving the conjecture.

Conjecture 14. *For any integer $t \geq 1$, t-TOUGH remains* NP*-hard for $(2t + 1)$-regular graphs.*

2.5 More open problems on complexity

There are still many interesting subclasses of graphs for which the complexity of recognizing t-tough graphs is unknown.

A number of these classes, all related to the complexity results obtained for regular graphs are given in [11]. These are repeated here, and they are based on a well-known long-standing conjecture in hamiltonian graph theory called Barnette's Conjecture, stating that every 3-connected, cubic, *planar*, bipartite graph is hamiltonian.

If any of the hypotheses in this conjecture is dropped, the conclusion that the graph is hamiltonian need not follow. Thus it seems interesting to consider the complexity of recognizing 1-tough graphs when one or more of the hypotheses in Barnette's Conjecture are dropped.

It is an easy exercise to prove that every 3-connected cubic graph is 1-tough. On the other hand, there are 2-connected, cubic, planar, bipartite graphs which are not 1-tough (see, e.g., [2]). The complexity of recognizing 1-tough graphs remains open for the following classes of graphs.

- 2-connected, cubic, planar, bipartite graphs;
- 2-connected, cubic, planar graphs;
- 2-connected, cubic, bipartite graphs;

- 2-connected, planar, bipartite graphs;
- 2-connected, planar graphs.

Tutte [85] has shown that every 4-connected planar graph is hamiltonian. On the other hand, there exist 3-connected, planar, bipartite graphs which are not 1-tough (e.g., the Herschel graph [[22], p. 472]). The complexity of recognizing 1-tough graphs remains open for the following classes.

- 3-connected, planar, bipartite graphs;
- 3-connected, planar graphs;
- 3-connected, bipartite graphs.

It is interesting to note that the complexity of recognizing hamiltonian graphs is known to be NP-hard for all of the above classes except possibly 3-connected, planar, bipartite graphs [1, 54].

As indicated above, we do not know the complexity of recognizing 1-tough, planar graphs. However, the next result might yield a clue. It follows from theorems in [40, 81].

Theorem 15. *Let G be a planar graph on at least 5 vertices. Then G is 4-connected if and only if $\omega(G - S) \leq |S| - 2$ for all vertex cuts $S \subset V(G)$ with $|S| \geq 3$.*

Since 4-connected graphs can be recognized in polynomial time, it follows that for planar graphs G, it can be determined in polynomial time whether $\omega(G-S) \leq |S| - 2$ for all vertex cuts $S \subset V(G)$ with $|S| \geq 3$. To determine if G is 1-tough, one needs to decide the similar inequality $\omega(G - S) \leq |S|$ for all vertex cuts $S \subset V(G)$. Perhaps this suggests that recognizing 1-tough, planar graphs can be done in polynomial time.

Dillencourt [39] has also inquired about the complexity of recognizing 1-tough, *maximal* planar graphs, noting that recognizing hamiltonian, maximal planar graphs is NP-hard. All we know is that there exist maximal planar graphs which are not 1-tough.

Another well-studied class of graphs for which the complexity of determining the toughness is open, is the class of chordal graphs. It is not difficult to observe that within this class of chordal graphs, studying tough sets can be restricted to sets that induce a connected graph. It is also an easy exercise to show that simplicial vertices do not belong to tough sets. Perhaps these observations can be used as a starting point for proving that the toughness of chordal graphs can be computed in polynomial time. If one would be able to prove polynomiality for the toughness and related vulnerability measures restricted to chordal graphs, these results would generalize results in [70]. The algorithms in [70] are the first to efficiently compute such measures for several nontrivial graph classes. In fact their approach is more widely applicable. Their algorithms compute two types of vectors which they call component number vectors and maximum component order vectors. These could be of interest for solving other vulnerability problems for the considered graph classes as well.

A related direction which could lead to NP-hardness proofs for new graph classes, is to find alternative proofs for NP-hardness of the original toughness problem. Related questions on (in)approximability seem to be totally unexplored and a good topic for future research.

3 Chvátal's conjectures

As noted earlier, being 1-tough is a necessary condition for a graph to be hamiltonian. In [34], Chvátal conjectured that there exists a finite constant t_0 such that every t_0-tough graph is hamiltonian. He showed in [34] that there exist $\frac{3}{2}$-tough nonhamiltonian graphs, and subsequently Thomassen [[20], p. 132] found t-tough nonhamiltonian graphs with $t > \frac{3}{2}$. Later results of Enomoto et al. [47] imply that there exist $(2 - \epsilon)$-tough nonhamiltonian graphs for arbitrary $\epsilon > 0$.

3.1 Toughness and hamiltonicity

For many years, the focus was on determining whether all 2-tough graphs are hamiltonian. One reason for this is that if all 2-tough graphs were hamiltonian, a number of important consequences [3] would follow. In addition, the results of Enomoto et al. [47] seemed to indicate that two might be the threshold for toughness that would imply hamiltonicity. The truth of the 2-tough conjecture would also imply the well-known result of Fleischner [52] that the *square* of any 2-connected graph is hamiltoninan. Moreover, it would imply the truth of two other conjectures that have been open for about thirty years: Every 4-connected line graph is hamiltonian [84], and every 4-connected claw-free graph is hamiltonian [75]. These conjectures are in fact equivalent, as later shown in [80]. However, it turns out that not all 2-tough graphs are hamiltonian. Indeed, we have the following result [8]. Here a graph is called *(non)traceable* if it does (not) admit a *Hamilton path*, i.e., a path containing all of its vertices.

Theorem 16. *For every $\epsilon > 0$, there exists a $(\frac{9}{4} - \epsilon)$-tough nontraceable graph.*

We believe that this result can be improved, and that we need a clever combination of new structural results and computer-aided search methods to find better examples, i.e., with a higher toughness. One reason for being hopeful is that the construction and building blocks we used to prove the above theorem are surprisingly simple at hindsight. We therefore continue this section by presenting a brief outline of the construction of these counterexamples, which were inspired by constructions in [3] and [16].

For a given graph H and $x, y \in V(H)$ we define the graph $G(H, x, y, \ell, m)$ as follows. Take m disjoint copies $H_1, \ldots, H_m$ of H, with x_i, y_i the vertices in H_i corresponding to the vertices x and y in H ($i = 1, \ldots, m$). Let F_m be the graph obtained from $H_1 \cup \ldots \cup H_m$ by adding all possible edges between pairs of vertices in $\{x_1, \ldots, x_m, y_1, \ldots, y_m\}$. Let $T = K_\ell$ and let $G(H, x, y, \ell, m)$ be the join $T \vee F_m$ of T and F_m.

The proof of the following theorem appeared in [8] and almost literally also in [3].

Theorem 17. *Let H be a graph and x, y two vertices of H which are not connected by a Hamilton path of H. If $m \geq 2\ell + 3$, then $G(H, x, y, \ell, m)$ is nontraceable.*

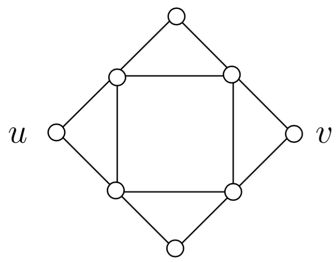

Figure 1: The graph L.

Consider the graph L of Figure 1. There is obviously no Hamilton path in L between u and v. Hence $G(L, u, v, \ell, m)$ is nontraceable for every $m \geq 2\ell + 3$. The toughness of these graphs was established in [8].

Theorem 18. *For $\ell \geq 2$ and $m \geq 1$,*

$$\tau(G(L, u, v, \ell, m)) = \frac{\ell + 4m}{2m + 1}.$$

Combining Theorems 17 and 18 for sufficiently large values of m and ℓ, one obtains the next result [8].

Corollary 19. *For every $\epsilon > 0$, there exists a $\left(\frac{9}{4} - \epsilon\right)$-tough nontraceable graph.*

It is easily seen from the proof in [8] that Theorem 17 remains valid if $m \geq 2\ell + 3$ and nontraceable are replaced by $m \geq 2\ell + 1$ and nonhamiltonian, respectively. Thus the graph $G(L, u, v, 2, 5)$ is a nonhamiltonian graph, which by Theorem 18 has toughness 2. This graph is sketched in Figure 2. It follows that a smallest counterexample to the 2-tough conjecture has at most 42 vertices. Similarly, a smallest nontraceable 2-tough graph has at most $|V(G(L, u, v, 2, 7))| = 58$ vertices.

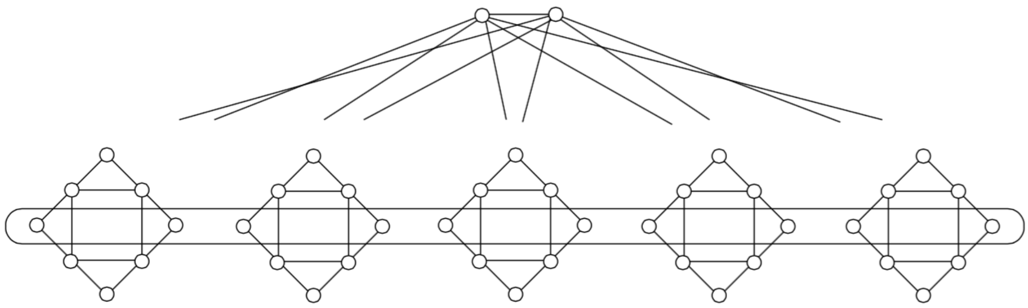

Figure 2: The graph $G(L, u, v, 2, 5)$.

A graph G is *neighborhood-connected* if the neighborhood of each vertex of G induces a connected subgraph of G. In [34], Chvátal also stated the following weaker version of the 2-tough conjecture: every 2-tough neighborhood-connected graph is hamiltonian. Since all counterexamples described above are neighborhood-connected, this weaker conjecture is also false.

Most of the ingredients used for constructing the above counterexamples were already present in [3]. It only remained to observe that using the specific graph L as a building block produced a graph with toughness at least 2. Note that the graph L itself does not have a particularly high toughness. One of the crucial properties of L is that the vertex cuts that contribute to the tough set of $G(L, u, v, \ell, m)$ are small and yield relatively few components. We hope that other building blocks and/or smarter constructions will lead to counterexamples with a higher toughness. This will probably require new structural results on clever combinations of suitable smaller graphs into larger nonhamiltonian ones with large toughness, together with directed computer search for suitable small graphs. Perhaps the concept of *weakly hamiltonian graphs* introduced by Chvátal (See, e.g. [35] for a nice exposition of structural results for nonhamiltonian graphs) will be a key ingredient to finding better constructions.

3.2 Toughness and other cycle structures

Constructions similar to those used to prove Theorem 16 have been used to establish other important results.

A *k-factor* of a graph G is a *spanning* k-regular subgraph of G, i.e., a k-regular subgraph that contains all the vertices of G. Hence, a Hamilton cycle is a special case of a 2-factor, namely a connected 2-factor.

Chvátal [34] obtained $\left(\frac{3}{2} - \epsilon\right)$-tough graphs without a 2-factor for arbitrary $\epsilon > 0$. These examples are all chordal. It was shown in [13] that every $\frac{3}{2}$-tough chordal graph has a 2-factor. Based on this, Kratsch [69] raised the question whether every $\frac{3}{2}$-tough chordal graph is hamiltonian. Using Theorem 17 in [8] it has been shown that this conjecture, too, is false.

Consider the graph M of Figure 3.

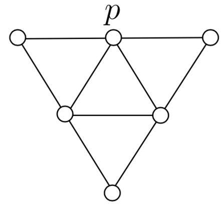

Figure 3: The graph M.

The graph M is chordal and has no Hamilton path with endvertices p and q. The graphs $G(M, p, q, \ell, m)$ are also chordal and, by Theorem 17, they are nontraceable whenever $m \geq 2\ell + 3$. By arguments similar to those used in the proof of Theorem 18, the toughness of $G(M, p, q, \ell, m)$ is $\dfrac{\ell + 3m}{2m + 1}$ if $\ell \geq 2$. Hence for $\ell \geq 2$ the graph $G(M, p, q, \ell, 2\ell + 3)$ is a chordal nontraceable graph with toughness $\dfrac{7\ell + 9}{4\ell + 7}$. This gives the following result from [8].

Theorem 20. *For every $\epsilon > 0$, there exists a $\left(\frac{7}{4} - \epsilon\right)$-tough chordal nontraceable graph.*

A *k-walk* in a graph G is a closed spanning walk of G that visits every vertex of G at most k times. Hence, a Hamilton cycle is a 1-walk. In [43], Ellingham and Zha used the same construction as above to give an infinite class of graphs of relatively high toughness without a k-walk. They obtained the following results.

Theorem 21. *Every 4-tough graph has a 2-walk.*

Theorem 22. *For every $\epsilon > 0$ and every $k \geq 1$ there exists a $\left(\frac{8k+1}{4k(2k-1)} - \epsilon\right)$-tough graph with no k-walk.*

To prove the latter theorem they first modified the graph L from Figure 1 and then relied on the same basic construction that was used in [8].

3.3 Chvátal's conjecture for special graph classes

Since Chvátal [34] introduced toughness in 1973, much research has been done that relates toughness conditions to the existence of cycle structures. But the most challenging of the conjectures he posed in [34] is still open: Is there a finite constant t_0 such that every t_0-tough graph is hamiltonian? If so, what is the smallest such t_0? We know from the above results that if the conjecture is true, then $t_0 \geq 9/4$.

Although the conjecture is still open for general graphs, we know that it is true for a number of well-studied graph classes, e.g., planar graphs, claw-free graphs and chordal graphs.

Since all 4-connected planar graphs are hamiltonian by a well-known theorem of Tutte [85], we have $t_0 > 3/2$ for planar graphs, and this result is best possible.

For a claw-free graph G we know that $\tau(G) = \kappa(G)/2$; consequently $t_0 \leq 7/2$ by a result of Ryjacek [80], combined with a result of Zhan [87] and Jackson [58] stating that all 7-connected line graphs are hamiltonian. This has recently been improved in [63] to 5-connected line graphs with minimum degree at least 6, so in particular to 6-connected line graphs. However Matthews and Sumner [75] have conjectured that 4-connected (2-tough) claw-free graphs are hamiltonian. For a survey on this and many related conjectures we refer to [26].

Let us turn again to chordal graphs, i.e., graphs that have no induced cycles of length greater than 3. Alternatively, one can view a chordal graph as the intersection graph of a family of subtrees of a tree. It was shown in [31] that every 18-tough graph on at least three vertices is hamiltonian, but this result is probably far from best possible. As we have just seen the best known negative result is from [8] where an infinite class of chordal graphs with toughness close to $7/4$ having no Hamilton path is constructed.

There are several subclasses of chordal graphs, however, for which the smallest toughness guaranteeing hamiltonicity is known. Recall that a graph is called a split graph if its vertex set can be partitioned into a clique and an independent set. Alternatively a split graph can be viewed as the intersection graph of a family of connected subgraphs of a star (and so split graphs are chordal graphs). It was shown in [71] that every $3/2$-tough split graph on at least three vertices is hamiltonian, and that this is best possible in the sense that there is a sequence $\{G_n\}_{n=1}^{\infty}$ of split graphs with no 2-factor and $\tau(G_n) \to 3/2$. This result was generalized by Kaiser, Král, and Stacho [62], who showed that $3/2$-tough *spiders* are hamiltonian; a spider is the intersection graph of a family of connected subgraphs of a subdivision of a star (and so spiders are chordal graphs). Keil [68] showed that every 1-tough *interval graph* is hamiltonian (an interval graph is the intersection graph of subpaths of a path), which is clearly best possible. Deogun, Kratsch and Steiner [38] generalized this by showing that 1-tough co-comparability graphs (not a subclass of chordal graphs) are hamiltonian.

In [27], toughness conditions are studied that guarantee the existence of a Hamilton cycle in *k-trees*. In this context, a k-tree is a graph that can be obtained from a K_k by repeatedly adding new vertices and joining them to a set of k mutually adjacent vertices. It is clear that a k-tree is a chordal graph. In [27], it is shown that every 1-tough 2-tree on at least three vertices is hamiltonian. This is generalized to a result on k-trees for $k \geq 2$ as follows: Let G be a k-tree. If G has toughness at least $(k+1)/3$, then G is hamiltonian. Moreover, infinite classes of nonhamiltonian 1-tough k-trees for each $k \geq 3$ are presented.

In a recent paper [25], inspired by a failed attempt to improve the result on chordal graphs due to Chen et al. [31], we were able to establish the truth of Chvátal's Conjecture for a superclass of split graphs we encountered before, namely for the class of $2K_2$-free graphs.

Theorem 23. *Every* 25*-tough* $2K_2$*-free graph on at least three vertices is hamiltonian.*

While this establishes Chvátal's Conjecture for a new graph class, like the result for chordal graphs [31], our bound is very likely to be far from extremal. It is conjectured in [77] that t-tough $2K_2$-free graphs with $t > 1$ are hamiltonian, but this seems extremely difficult to prove. The proof of Theorem 23 in [25] relies on the very restrictive structure of *triangle-free* $2K_2$-free graphs, and we were able to prove a sharp result for such graphs, namely that triangle-free $2K_2$-free graphs are hamiltonian if and only if they are 1-tough.

A natural question, in light of the disproof of the 2-tough conjecture for general graphs, is what minimum level of toughness will ensure that a graph from a restricted graph class is hamiltonian. More specifically, are 2-tough chordal graphs hamiltonian? Are 2-tough $2K_2$-free graphs hamiltonian?

What about triangle-free graphs? Are 2-tough triangle-free graphs hamiltonian? It is conjectured in [12] that for all $\epsilon > 0$, there exists a $(2-\epsilon)$-tough triangle-free graph that does not even contain a 2-factor. An infinite collection of triangle-free graphs are given there that clearly have no 2-factor. It appears that the toughness of these graphs approaches 2 as the order $n \to \infty$. However, establishing the toughness appears difficult. On the other hand, Ferland [51] has found an infinite class of nonhamiltonian triangle-free graphs whose toughness is at least 5/4. Note that the toughness of the Petersen graph is 4/3. However, the Petersen graph is not an infinite class.

The following result appeared in [4] and shows that Chvátal's Conjecture that there exists a finite constant t_0 such that all t_0-tough graphs are hamiltonian is true within the class of graphs having $\delta(G) \geq \epsilon n$, for any fixed $\epsilon > 0$.

Theorem 24. *Let G be a t-tough graph on $n \geq 3$ vertices with $\delta > n/(t+1) - 1$. Then G is hamiltonian.*

In [72], we have tried to characterize all graphs H such that every 1-tough H-free graph on at least three vertices is hamiltonian, following up on earlier work by Jung [60] and Nikoghosyan [77]. We almost established a full characterization, leaving just one open case. We proved the following two results in [72]. Here $H \cup F$ denotes the disjoint union of two vertex-disjoint graphs H and F, and we use the shorthand notation $H \cup F$-free instead of $(H \cup F)$-free.

Theorem 25. *Let R be an induced subgraph of P_4, $K_1 \cup P_3$ or $2K_1 \cup K_2$. Then every R-free 1-tough graph on at least three vertices is hamiltonian.*

The case with $K_1 \cup P_3$ was independently proved in [77], where the case with P_4 was conjectured, and the nonhamiltonian $K_1 \cup K_2$-free graphs were characterized. The case with P_4 has been proved back in the 1970s [60], where P_4-free graphs were studied as D^*-graphs, but they are more commonly known as cographs (since the complement of a P_4-free graph is also P_4-free).

Theorem 26. *Let R be a graph on at least three vertices. If every R-free 1-tough graph on at least three vertices is hamiltonian, then R is an induced subgraph of $K_1 \cup P_4$.*

Note that every induced subgraph of $K_1 \cup P_4$ is either $K_1 \cup P_4$ itself, or an induced subgraph of P_4, $K_1 \cup P_3$ or $2K_1 \cup K_2$. By the above two theorems, the only graph for which we do not know whether forbidding it can ensure a 1-tough graph to be hamiltonian is $K_1 \cup P_4$. We repeat this here as an open problem, but it appeared as a conjecture in [77].

Question 27. *Is every $K_1 \cup P_4$-free 1-tough graph on at least three vertices hamiltonian?*

This question seems to be very hard to answer, even if we impose a higher toughness. In the light of the discussion, it is interesting to consider the following weaker version of Question 27.

Question 28. *Is the general conjecture of Chvátal true for $K_1 \cup P_4$-free graphs?*

Some of the partial results in [25] can be proved for $K_1 \cup P_4$-free graphs as well, but a similar approach fails for solving the general problem.

3.4 Toughness and factors

In [34], Chvátal also conjectured that every k-tough graph on $n \geq k + 1$ vertices and kn even contains a k-factor. Enomoto et al. [47] gave a decisive answer to Chvátal's conjecture in the following two theorems.

Theorem 29. *Let G be a k-tough graph on n vertices with $n \geq k + 1$ and kn even. Then G has a k-factor.*

Theorem 30. *Let $k \geq 1$. For every $\epsilon > 0$, there exists a $(k - \epsilon)$-tough graph G on n vertices with $n \geq k + 1$ and kn even which has no k-factor.*

In particular, every 2-tough graph contains a 2-factor, and for every $\epsilon > 0$, there exist infinitely many $(2 - \epsilon)$-tough graphs with no 2-factor.

In [44], Enomoto strengthened Theorem 29.

Theorem 31. *Let k be a positive integer and G be a graph on n vertices with $n \geq k + 1$ and kn even. Suppose $|S| \geq k \cdot \omega(G - S) - \frac{7k}{8}$ for all $S \subseteq V$ with $\omega(G - S) \geq 2$. Then G has a k-factor.*

We think these structural results and their improvements go beyond the common interest of the readership of this bulletin, so we decided to complete this section with some general remarks and references to related work. We apologize to the interested reader for the inconvenience.

Further improvements on Theorem 31 and related results were obtained in [45], [46], [29], [64], and [30]. Other related results involving sufficient minimum degree conditions for a t-tough graph to contain a 2-factor and 3-factor appeared in [16] and [15], respectively. In [42] some of the above results have been extended to connected factors. In [65] it was shown that a 1-tough bipartite graph on $n \geq 3$ vertices has a 2-factor. In [49], the authors present a minimum degree condition for a 1-tough graph to have a 2-factor with a specific number of cycles.

A number of results on factors have appeared relating toughness to (r, k)-factor-critical graphs. A graph G is (r, k)-*factor-critical* if $G - X$ contains an r-factor for all $X \subseteq V$ with $|X| = k$. For $r \geq 2$, these graphs were studied in [73] under the name (r, k)-extendable graphs. Later results appeared in [28, 45, 50, 82].

Other related results, including an interesting edge variant on toughness, appeared in [66, 67].

4 Final remarks

Since Chvátal [34] introduced toughness in 1973, much research has been done that relates toughness conditions to the existence of cycle structures. In this paper, we have mainly focussed on results and open problems with an algorithmic flavor. Much more material on structural results can be found in [7]. We will list some of the most challenging open problems that we encountered before and also briefly mention some of the others that we have not discussed.

From the current exposition we have seen that research on toughness has also focused on computational complexity issues. In particular, we know that recognizing t-tough graphs is NP-hard in general, whereas it is polynomial within the class of claw-free graphs and within the class of $2K_2$-free graphs, including split graphs. For many other interesting classes, this complexity question is still open, e.g., for (maximal) planar graphs and for chordal graphs. Within the class of r-regular graphs with $r \geq 3t$, recognizing t-tough graphs has been shown to be NP-hard. The problem is trivial if $r < 2t$, but its complexity is open for values of r with $2t \leq r < 3t$. It was conjectured in [55] to be polynomial for $r = 2t$, and seems especially interesting when $r = 2t + 1$.

Historically, the motivation for most of the presented research was based on a number of conjectures in [34]. The most challenging of these conjectures is still open: Is there a finite constant t_0 such that every t_0-tough graph is hamiltonian?

Although the conjecture is still open for general graphs, we know that it is true for a number of well-studied graph classes, e.g., planar graphs, claw-free graphs and chordal graphs, and also for $2K_2$-free graphs. The gaps in our knowledge about the smallest value of t_0 for claw-free, chordal and $2K_2$-free graphs imply a number of challenging open problems. The same is true for the class of triangle-free graphs. It is known that there exists an infinite class of $5/4$-tough triangle-free nonhamiltonian graphs [51], and it even appears that a class of triangle-free graphs with no 2-factor constructed in [12] has toughness approaching 2 from below. These examples suggest the intriguing possibility that every 2-tough triangle-free graph is hamiltonian, though it remains completely open whether the t_0-tough conjecture holds for the class of triangle-free graphs.

Suppose we also impose a minimum degree condition. The examples that disproved the 2-tough conjecture all have $\delta = 4$. On the other hand we know that if G is a 2-tough graphs on n vertices with $\delta \geq n/3$, then G is hamiltonian. What if $5 \leq \delta < n/3$? The early research on toughness and cycle structure concentrated on sufficient degree conditions which, combined with a certain level of toughness, would yield the existence of long cycles. The survey [7] contains a wealth of results in this direction. One of the major open problems in this area is the conjecture that every 1-tough graph on n vertices with $\sigma_3 \geq n \geq 3$ has a cycle of length at least $\min\{n, (3n+1)/4 + \sigma_3/6\}$. Here σ_3 is the minimum degree sum of three mutually nonadjacent vertices. Another interesting problem is to find the best possible minimum degree condition to ensure that a 1-tough triangle-free graph is hamiltonian. We know the answer lies somewhere between $(n+2)/4$ and $(n+1)/3$.

If we do not impose a degree condition, toughness conditions can still guarantee cycles of length proportional to a function of the number of vertices of the graph. Two of the most challenging open problems in this area are whether there exist positive constants A and B, depending only on t, such that every 2-connected, respectively 3-connected, t-tough graph on n vertices has a cycle of length at least $A \log n$, respectively n^B. Both problems have affirmative solutions for planar graphs.

Another area of research has involved finding toughness conditions for the existence of certain factors in graphs. Whereas Chvátal's original conjecture on the existence of k-factors turned out to be true, one of the challenging remaining open problems in this area is to determine whether every $3/2$-tough maximal planar graph has a 2-factor. If so, are they all hamiltonian? We also do not know whether a $3/2$-tough planar graph has a 2-factor.

To complete this paper, let us consider one more problem area with an algorithmic flavor that is remotely related to the content and that might be of interest to the readership of the bulletin.

There are several results in (hamiltonian) graph theory of the form $\mathcal{P}_1$ implies $\mathcal{P}_2$, where $\mathcal{P}_1$ is an NP-hard property of graphs and $\mathcal{P}_2$ is an NP-hard (cycle) structure property, and one might wonder about the practical value of such theorems. Two such theorems are the well-known theorems of Chvátal and Erdös [36] and Jung [61]. In [35], Chvátal gave a proof of the Chvátal-Erdös Theorem [36] which constructs in polynomial time either a Hamilton cycle in a graph G or an independent set of more than $\kappa(G)$ vertices in G. In [5], the authors provided a similar type of polynomial time constructive proof for Jung's Theorem [61] on graphs with at least 16 vertices. It is possible that other theorems in graph theory with an NP-hard hypothesis and an NP-hard conclusion also have polynomial time constructive proofs.

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