Reflections on Influential Scientists and Ideas
A new section of the Bulletin of the EATCS

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The piece by David Avis (School of Informatics, Kyoto University) that you are about to read offers a look at George Dantzig's best known contribution, the *simplex method*, and at its connections with theoretical computer science. In this article, David Avis also provides some food for thought for our research community and argues for a collaboration of the TCS and optimization communities to settle the question of whether there is a polynomial time pivot selection rule for the time-honoured simplex method.

Last year marked the centenary of George Bernard Dantzig's birth, and Kazuo Iwama originally commissioned this reflection piece to David Avis to celebrate that anniversary. Upon receiving David's article, Kazuo and I were struck by the thought that readers of the Bulletin might enjoy reading short articles devoted to anniversaries of influential scientists and to ideas related to theoretical computer science at large. Such reflection pieces could provide a new look at the legacy of pioneers and at some of the pearls of our subject, as well as possibly highlight some of the challenges that still need to be met. We feel that they would be useful to young members of our community, students and experienced researchers alike.

A list of upcoming anniversaries includes the 200th anniversary of the birth of George Boole and the 100th anniversary of the birth of Richard Hamming in 2015, as well as the centenary of Claude Shannon in 2016. Please get in touch with Kazuo if you are interested in contributing a reflection piece on any of those figures, or on a scientist or a "pearl of theoretical computer science" of your choice.

For the moment, Kazuo and I hope that you will enjoy David Avis' piece that gives this new section of the Bulletin an excellent start.

GEORGE DANTZIG: FATHER OF THE SIMPLEX METHOD

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100 years have passed since George Bernard Dantzig was born, about 70 years since he started developing the simplex method, and 10 since he died. His main legacy is the simplex method, which *Computer Science and Engineering* included as one of the top 10 algorithms (sic) of the 20th century. In 2006 Martin Grötschel said: "the development of linear programming is - in my opinion - the most important contribution of the mathematics of the 20th century to the solution of practical problems arising in industry and commerce." Yet linear programming, and especially the simplex method, has had a tenuous relation to theoretical computer science (TCS) over the years.

Let us begin with the fact that it is the *simplex method* not the *simplex algorithm*. It is a method because it describes a class of algorithms. An algorithm in the class is initialized at any vertex of a convex polyhedron and will follow edges on the boundary of the polyhedron until reaching a vertex that maximizes a given linear function[1]. These algorithms are specified by a *pivot rule* that is normally deterministic and defines a unique path along which the objective function increases monotonically until an optimum vertex is reached. It may not be the case that the algorithm makes progress along an edge at every step: it may make repeated pivots at a given vertex before making progress, a phenomenon known as *stalling*. Dantzig's original (and still widely used) pivot rule has two unfortunate properties. Firstly, it may stall indefinitely, going into an infinite loop. Secondly, it may follow a path on the polyhedron of exponential length, as shown by Klee and Minty in 1972. The first problem can be efficiently solved by the lexicographic ratio test, a method that simulates simplicity and renders all pivot rules finite, or by perturbation. The second problem has never been solved. Following Klee and Minty, a series of papers gave exponential lower bounds for the then known pivot rules, using variants of Klee-Minty cubes. These types of examples can be defeated by history based rules, such as those introduced by Zadeh in 1980. These

[1]For simplicity we omit the cases where the polyhedron is unbounded or empty.

rules resisted analysis for more than 30 years, despite Zadeh's offer of a $1000 prize.

Even with these serious limitations, the simplex method dominates optimization, especially integer programming, where it is routinely used to optimally solve large examples of NP-hard integer programming problems. For example the Concorde program of Applegate, Bixby, Cook and Chvátal has found the optimum solution of traveling salesman problems (TSPs) with as many as 85,900 cities. It is based on the branch-and-cut method that generates enormous numbers of extremely large linear programs. The 85,900 city TSP involved the solution of about one million sparse LPs each with roughly 100,000 constraints and 170,000 variables. Each new LP is created by adding a number of cutting planes to an LP with a fractional optimum solution. Using Dantzig's dual simplex method and the original optimum solution, this new LP can be optimized with ease in practice, but of course there is no theoretical foundation for this. Incidentally this method was pioneered by Dantzig, along with Fulkerson and Johnson, in their groundbreaking 1954 paper where they optimally solved a 49 city TSP by hand[2]!

Other methods for linear programming exist of course. Starting with the ellipsoid method of Khachian in 1979 the competing class of interior point algorithms has been extensively developed. To a TCS eye these are eminently preferable, being provably polynomial. Although the ellipsoid method is a non-starter for solving any problem encountered in practice, later algorithms are competitive with the simplex method. However this should not ease TCS discomfort with the simplex method. Imagine things were reversed: Dantzig had discovered an interior method in 1947 and Khachian had proposed the simplex method in 1979. What chance would Khachian have had of getting a non-finite, non-algorithm running in exponential time for a problem in *P* accepted to FOCS/STOC/SODA? Would it even have been programmed? When it comes to handling cutting planes the simplex method wins hands down against interior point methods. And it is integer programming that provides the huge bulk of LPs that need to be solved in practice. The simplex method also delivers an optimal basis and a proof of optimality via the dual multipliers that can be independently verified. But most uncomfortably of all: it really does work in practice and seemingly flies in the face of a lot of what TCS teaches students about *P*, *NP*-hardness, and exponential algorithms.

Given its importance why is it that Dantzig's simplex method was largely ignored in the TCS community until relatively recently? Most of us who studied at Stanford's Department of Operations Research, that Dantzig pioneered, largely funded, and presided over, were not enamored by the rather opaque description

[2]Despite Michigan State computer scientist Randy Olson's recent claims to the contrary: "With 50 landmarks to put in order, we would have to exhaustively evaluate 3 x 10^{64} possible routes to find the shortest one." http://www.randalolson.com/2015/03/08/computing-the-optimal-road-trip-across-the-u-s/)

in his opus *Linear Programming and Extensions*. This may have delayed the widespread understanding of the simplex method but was completely rectified by Chvátal's lucid description in the first few chapters of his now classic *Linear Programming*. I always found it very impressive that my PhD supervisor, a 27 year old nontenured assistant professor in the department at the time, dared to rewrite Dantzig's description in plain English[3]. Anyway he did, so there is no excuse for anyone not to understand it.

Many TCS books on algorithms ignore linear programming altogether. Where it is discussed it is often treated almost as a footnote. The encyclopedic Cormen, Leiserson, Rivest and Stein's *Algorithms(3rd edition)* describes it in Chapter 29 (of 35), Selected Topics. The excellent and highly readable Kleinberg and Tardos' *Algorithm Design* treats it in Chapter 11 (of 13), a chapter on approximation algorithms! This despite the fact that both texts contain many examples of the simplex method in disguised form in earlier chapters: Dijkstra's algorithm, Bellman-Ford, network flows, matchings etc.

Early TCS interest in linear programming was shown by computational geometers. However algorithms that are exponential in the dimension and linear in the number of constraints are of little interest when the number of variables is counted in the hundreds of thousands. A major theoretical result appeared in 1991 with the joint discovery by Kalai and Matousek, Sharir & Welzl of a subexponential pivot selection method based on selecting random facets at the current vertex. However this result has never been derandomized and, being recursive in the dimension, is not practical for the kinds of LPs encountered in practice. Another line of research involved the probabilistic analysis of the simplex method, commencing in the late 1970s with Borgwardt(Lanchester prize) and continuing into this century with the smoothed analysis of Spielman and Teng(Gödel prize). This is deep work indeed and gives considerable insight into the success of the simplex method for certain types of problems. However large scale combinatorial problems hardly seem to fit these models. More recently, in 2011, Friedmann, Hansen & Zwick gave subexponential lower bounds for the random facet rule and also for the more intuitive random edge rule, receiving STOC's best paper award. Also in 2011, Friedmann gave a subexponential lower bound for Zadeh's history based rule, solving that 30 year open problem, and picked up a cheque for $1000 from the man in person at an IPAM meeting in Los Angeles. Similar results followed. Each simplex pivot generates a lot of information about the polyhedron, and for a polynomial time pivot rule this entire history could be recorded. So history based rules offer good candidates for polynomial upper bounds. Much work needs to be done here. Surely the close collaboration of TCS and the optimization commu-

[3]Earlier Dantzig had asked Chvátal how old he was. When he heard, Dantzig replied: "Then I pity you because you are going to have to live through all the shit that is coming up."

nity would be able to settle this question: is there or is there not a polynomial time pivot selection rule for the simplex method? Of course I think all of us, including George, hope for a positive answer that is both strongly polynomial time and a winner in practice!

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