Ground Term Rewriting

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Abstract

We study the notion of a stub equality for a congruence generated by a ground term rewrite system (GTRS). We study the congruence generated by the union of GTRSs R and S, where the congruences generated by R and S intersect with respect to their stubs. We show that for any equivalent reduced GTRSs R and S, the same number of terms appear as subterms in R as in S. We give an upper bound on the number of reduced GTRSs equivalent to a given reduced GTRS R. We show that for any convergent GTRS R, one can construct an equivalent reduced GTRS V such that $\rightarrow_V \subseteq \rightarrow_R^*$.

keywords: ground tem rewrite system; bottom-up tree automaton

1 Introduction

Ground term rewrite systems have been studied by numerous researchers, see [1]-[24]. We abbreviate the expression ground term rewrite system by GTRS. Snyder [18] introduced and studied the concept of a reduced GTRS. He [18] gave a fast algorithm for generating a reduced GTRS equivalent to a given GTRS. His method also generates all reduced GTRSs equivalent to a given GTRS. He [18] showed that any equivalent reduced GTRSs R and S consist of the same number of rewrite rules. He [18] also showed that for a GTRS R consisting of n rules, there are at most 2^n reduced GTRSs equivalent to R.

Let ρ be a congruence over the term algebra **TA**. We study $stub(\rho)$, which is a set of ρ classes. In the special case, when $\rho = \leftrightarrow_R^*$ for some reduced GTRS R, $stub(\leftrightarrow_R^*)$ is equal to the set of the $\leftrightarrow_R^*$-classes of the terms appearing as subterms in R.

*This research was partially supported by the TÁMOP-4.2.2/08/1/2008-0008 program of the Hungarian National Development Agency.

We introduce the notion of a stub equation for ρ. Intuitively, a stub equation for ρ is of the form $f(Z_1, \ldots, Z_m) \approx Z$, where $f \in \Sigma_m$, $m \geq 0$, $Z_1, \ldots, Z_m, Z \in \mathit{stub}(\rho)$, and $f^{\mathbf{TA}/\rho}(Z_1, \ldots, Z_m) = Z$. $STN(\rho)$ stands for the set of all stub equations for ρ. $\mathit{stub}(\rho)$ and $STN(\rho)$ uniquely describe the congruence relation ρ. For a given reduced GTRS R, we can effectively construct $\mathit{stub}(\rho)$ and $STN(\rho)$.

Let ρ and τ be congruences over the term algebra $\mathbf{TA}$. We say that ρ and τ *intersect with respect to their stubs* if the following holds. For any $Z_1 \in \mathit{stub}(\rho)$ and $Z_2 \in \mathit{stub}(\tau)$, $Z_1 \cap Z_2 = \emptyset$ or $Z_1 = Z_2$. For example, let $\Sigma = \Delta \cup \Gamma$ and $\Delta \cap \Gamma = \emptyset$. Consider the GTRSs R and S over Σ, where $R \subseteq T_\Delta \times T_\Delta$, and $S \subseteq T_\Gamma \times T_\Gamma$. Then for any $Z_1 \in \mathit{stub}(\leftrightarrow_R^*)$ and $Z_2 \in \mathit{stub}(\leftrightarrow_S^*)$, $Z_1 \cap Z_2 = \emptyset$. Thus $\leftrightarrow_R^*$ and $\leftrightarrow_S^*$ intersect with respect to their stubs.

We show the following results. For any GTRSs R and S over a ranked alphabet Σ, we can decide whether $\leftrightarrow_R^*$ and $\leftrightarrow_S^*$ intersect with respect to their stubs. Furthermore, for any GTRSs R and S over a ranked alphabet Σ, the following three conditions are pairwise equivalent.

(a) $\leftrightarrow_R^*$ and $\leftrightarrow_S^*$ intersect with respect to their stubs.

(b) $STN(\leftrightarrow_{R \cup S}^*) = STN(\leftrightarrow_R^*) \cup STN(\leftrightarrow_S^*)$.

(c) $\mathit{stub}(\leftrightarrow_{R \cup S}^*) = \mathit{stub}(\leftrightarrow_R^*) \cup \mathit{stub}(\leftrightarrow_S^*)$.

We study the congruence $\leftrightarrow_{R1 \cup \cdots \cup Rn}^*$, where $R1, R2, \ldots, Rn$, $n \geq 2$, are GTRSs and any two of $\leftrightarrow_{R1}^*, \ldots, \leftrightarrow_{Rn}^*$ intersect with respect to their stubs.

We show some elementary properties of reduced GTRSs on the basis of the results of Snyder [18] and of Fülöp and Vágvölgyi [10]. We show that for any equivalent reduced GTRSs R and S, the same number of terms appear as subterms in R as in S. We present some simple correspondences between a reduced GTRS R and the algebraic constructs associated with the congruence $\leftrightarrow_R^*$. We give an upper bound on the number of reduced GTRSs equivalent to a given reduced GTRS R. This upper bound is less than or equal to that of Snyder [18]. Finally we show that for any convergent GTRS R, one can effectively construct an equivalent reduced GTRS V such that $\rightarrow_V \subseteq \rightarrow_R^*$.

In Section 2 we recall the notations and concepts to be used. In Section 3, we adopt and study some basic algebraic constructs associated with GTRSs. In Sections 4 - 6 we present our main results. The examples of Section 7 help the reader understand our concepts and results.

2 Preliminaries

In this section we present a brief review of the notions, notations, and preliminary results used in the paper. We illustrate the concepts and results of this and the forthcoming sections by the examples of Section 7.

Sets and Relations. The cardinality of a set A is denoted by $card(A)$. Let $\rho \subseteq A \times A$ be a binary relation on a set A. We denote by ρ^* the reflexive, transitive closure of ρ.

Let ρ be an equivalence relation on A. Then for every $a \in A$, we denote by $[a]_\rho$ the ρ-class containing a, i.e. $[a]_\rho = \{ b \mid a\rho b \}$. Let H be a set of ρ-classes, then $\bigcup H = \bigcup(Z \mid Z \in H)$.

Terms. A ranked alphabet Σ is a finite set of symbols in which every element has a unique rank in the set of nonnegative integers. For each integer $m \geq 0$, Σ_m denotes the elements of Σ which have rank m.

Let Y be a set. The set of terms over Σ with variables in Y is the smallest set U for which

(i) $\Sigma_0 \cup Y \subseteq U$ and

(ii) $f(t_1, \ldots, t_m) \in U$ whenever $f \in \Sigma_m$ with $m \geq 1$ and $t_1, \ldots, t_m \in U$.

For each $f \in \Sigma_0$, we mean f by $f()$. Terms are also called trees. The set $T_\Sigma(\emptyset)$ is written simply as T_Σ and called the set of ground trees over Σ.

We need a countably infinite set $X = \{ x_1, x_2, \ldots \}$ of variable symbols kept fixed throughout the paper. The set of the first n elements $x_1, \ldots, x_n$ of X is denoted by X_n. The set $T_\Sigma(\emptyset)$ is written simply as T_Σ and called the set of ground trees over Σ. For each $n \geq 1$, we define the subset $C_\Sigma(X_n)$ of $T_\Sigma(X_n)$ as follows. A tree $t \in T_\Sigma(X_n)$ is in $C_\Sigma(X_n)$ if and only if each variable symbol of X_n appears exactly once in t. For example, if $\Sigma = \Sigma_0 \cup \Sigma_2$ with $\Sigma_0 = \{ \# \}$ and $\Sigma_2 = \{ f \}$, then $f(x_1, f(\#, x_1)) \in T_\Sigma(X_1)$ but $f(x_1, f(\#, x_1)) \notin C_\Sigma(X_1)$. Furthermore, $f(x_2, f(\#, x_1)) \in C_\Sigma(X_2)$. The elements of $C_\Sigma(X_n)$ are called *contexts*.

The notion of *tree substitution* is defined as follows. Let $m \geq 0$, $p \in T_\Sigma(X_m)$ and $t_1, \ldots, t_m \in T_\Sigma$. We denote by $p[t_1, \ldots, t_m]$ the tree which is obtained from p by replacing each occurrence of x_i in t by t_i for every $1 \leq i \leq m$.

For a tree $t \in T_\Sigma(Y)$, the height $height(t)$ and the set $sbt(t)$ of *subtrees* of t is defined by tree induction.

(i) If $t \in \Sigma_0 \cup Y$, then $height(t) = 0$ and $sbt(t) = \{t\}$.

(ii) If $t = f(t_1, \ldots, t_n)$ with $f \in \Sigma_n$, $n > 0$, then $height(t) = 1 + \max\{height(t_i) \mid 1 \leq i \leq n\}$ and $sbt(t) = \{ t \} \cup (\cup_{i=1}^{n} sbt(t_i))$.

For a tree language $L \subseteq T_\Sigma$, the set $sbt(L)$ of subtrees of elements of L is defined by the equality $sbt(L) = \bigcup(sbt(t) \mid t \in L)$. We say that L is *closed under subtrees* if $sbt(L) \subseteq L$.

For any $f \in \Sigma_1$ and $t \in T_\Sigma$,
(i) $f^0(t) = t$, and
(ii) $f^n(t) = f(f^{n-1}(t))$ for $n \geq 1$.

Algebras. Let Σ be a ranked alphabet. A Σ algebra is a system $\mathbf{B} = (B, \Sigma^{\mathbf{B}})$, where B is a nonempty set, called the carrier set of $\mathbf{B}$, and $\Sigma^{\mathbf{B}} = \{ f^{\mathbf{B}} \mid f \in \Sigma \}$ is a Σ-indexed family of operations over B such that for every $f \in \Sigma_m$ with $m \geq 0$, $f^{\mathbf{B}}$ is a mapping from B^m to B. An equivalence relation $\rho \subseteq B \times B$ is a congruence on $\mathbf{B}$ if

$$f^{\mathbf{B}}(t_1, \ldots, t_m) \rho f^{\mathbf{B}}(p_1, \ldots, p_m)$$

whenever $f \in \Sigma_m$, $m \geq 0$, and $t_i \rho p_i$, for $1 \leq i \leq m$. For each $B' \subseteq B$, let $[B']_\rho = \{ [b]_\rho \mid b \in B' \}$. The least congruence on $\mathbf{B}$ containing a given relation $\sigma \subseteq B \times B$ is called the *congruence generated by* σ. A congruence on $\mathbf{B}$ is finitely generated if it is generated by a finite relation $\sigma \subseteq B \times B$. We define the *quotient algebra* $\mathbf{B}/\rho = ([B]_\rho, \Sigma^{\mathbf{B}/\rho})$ of the algebra $\mathbf{B}$ modulo the congruence ρ as follows. For all $f \in \Sigma_m$, $m \geq 0$, and $b_1, \ldots, b_m$, we put $f^{\mathbf{B}/\rho}([b_1]_\rho, \ldots, [b_m]_\rho) = [f^{\mathbf{B}}(b_1, \ldots, b_m)]_\rho$.

In this paper we shall mainly deal with the algebra $\mathbf{TA} = (T_\Sigma, \Sigma)$ of terms over Σ, where for any $f \in \Sigma_m$ with $m \geq 0$ and $t_1, \ldots, t_m \in T_\Sigma$, we have

$$f^{\mathbf{TA}}(t_1, \ldots, t_m) = f(t_1, \ldots, t_m)\,.$$

We adopt the concepts of a simple class and of a compound class of a congruence ρ on the term algebra $\mathbf{TA}$ from [10]. Let ρ be a congruence on $\mathbf{TA}$. A ρ-class Z is called simple if for any function symbols $f \in \Sigma_m, g \in \Sigma_n$, with $m, n \geq 0$ and ρ-classes $Z_1, \ldots, Z_m, W_1, \ldots, W_n$, if $f^{\mathbf{TA}/\rho}(Z_1, \ldots, Z_m) = Z$ and $g^{\mathbf{TA}/\rho}(Z_1, \ldots, Z_n) = Z$, then $f = g$, $m = n$, $Z_1 = W_1, \ldots, Z_m = W_m$. If a ρ-class Z is not simple then it is called a compound class. The set of all simple classes is denoted by $simp(\rho)$. The set of all compound classes is denoted by $comp(\rho)$.

Next we adopt the trunk of a congruence ρ from [10]. Let ρ be a congruence on $\mathbf{TA}$, the trunk $trunk(\rho)$ of ρ is the set $sbt(\bigcup comp(\rho))$. By direct inspection of the definition of $trunk(\rho)$, we get the following.

Proposition 2.1. *For any congruence ρ on* **TA**, *$trunk(\rho)$ is closed under subtrees.*

We write $stub(\rho)$ for $[trunk(\rho)]_\rho$. Obviously, $trunk(\rho) = \bigcup stub(\rho)$. A $stub(\rho)$ *equality* is of the form

$$f^{\mathbf{TA}/\rho}(Z_1, \ldots, Z_m) = Z\,, \tag{1}$$

where $Z \in stub(\rho)$.

Lemma 2.2. *Let ρ be a congruence on* **TA**. *For any $stub(\rho)$ equality*

$$f^{\mathbf{TA}/\rho}(Z_1, \ldots, Z_m) = Z\,,$$

we have $Z_1, \ldots, Z_m \in stub(\rho)$.

Proof. Let $t_i \in Z_i$ for $i = 1, \ldots, m$. Then $f(t_1, \ldots, t_m) \in Z$. Thus $f(t_1, \ldots, t_m) \in$ *trunk*(ρ). By Proposition 2.1, $t_1, \ldots, t_m \in$ *trunk*(ρ). Consequently, $Z_1, \ldots, Z_m \in$ *stub*(ρ). □

We say that the *stub*(ρ) equality (1) is a *comp*(ρ) equality if Z is a compound ρ-class. The set of all stub equalities for ρ is denoted by $STY(\rho)$. The set of all compound equalities for ρ is denoted by $COY(\rho)$. When ρ is apparent from the context, we write stub equality rather than *stub*(ρ) equality, we write compound equality rather than *comp*(ρ) equality, we write STY rather than $STY(\rho)$, and we write COY rather than $COY(\rho)$. Apparently, $COY \subseteq STY$.

Consider the ranked alphabet $\Sigma \cup$ *stub*(ρ), where the elements of *stub*(ρ) are considered as nullary symbols. We represent the stub equality (1) by the pair

$$f(Z_1, \ldots, Z_m) \approx Z\,, \tag{2}$$

of terms over $T_{\Sigma \cup stub(\rho)}$. We call (2) a *stub*($\rho$) equation. We say that the *stub*(ρ) equation (2) is a *comp*(ρ) equation if Z is a compound ρ-class. $STN(\rho)$ is the set of *stub*(ρ) equations. Similarly, $CON(\rho)$ is the set of *comp*(ρ) equations. When ρ is apparent from the context, we write stub equation rather than *stub*(ρ) equation, we write compound equation rather than *comp*(ρ) equation, we write STN rather than $STN(\rho)$, and we write CON rather than $CON(\rho)$. Apparently, $CON \subseteq STN$.

Definition 2.3. Let ρ and τ be congruences over the term algebra **TA**. We say that ρ and τ *intersect with respect to their stubs* if the following holds. For any $Z_1 \in$ *stub*(ρ) and $Z_2 \in$ *stub*(τ), $Z_1 \cap Z_2 = \emptyset$ or $Z_1 = Z_2$.

Apparently, ρ and ρ intersect with respect to their stubs. If ρ and τ intersect with respect to their stubs, then τ and ρ intersect with respect to their stubs. We now give a ranked alphabet Σ such that intersecting with respect to their stubs is not a transitive relation on the congruences over the term algebra $\mathbf{TA} = (T_\Sigma, \Sigma)$. Let $\Sigma = \Sigma_0 = \{a, b, c, d, e\}$. We define the congruences ρ, τ, and ω over the term algebra **TA**. ρ has two congruence classes: $\{a, b, c\}$ and $\{d, e\}$. τ has four congruence classes: $\{a\}$, $\{b\}$, $\{c\}$, and $\{d, e\}$. ω has three congruence classes: $\{a, b\}$, $\{c\}$, and $\{d, e\}$. Then *comp*(ρ) = *stub*(ρ) consists of two classes: $\{a, b, c\}$ and $\{d, e\}$. *comp*(τ) = *stub*(τ) consists of one class: $\{d, e\}$. *comp*(ω) = *stub*(ω) consists of two classes: $\{a, b\}$ and $\{d, e\}$. ρ and τ intersect with respect to their stubs, and τ and ω intersect with respect to their stubs. However, ρ and τ do not intersect with respect to their stubs.

Ground Term Rewrite Systems. A *ground term rewrite system* (GTRS) *over a ranked alphabet* Σ is a finite subset R of $T_\Sigma \times T_\Sigma$. The elements of R are called rules and a rule $(l, r) \in R$ is written in the form $l \to r$ as well. Moreover, we say that l is the left-hand side and r is the right-hand side of the rule $l \to r$. The elements of R can be used only in one direction given by the system to define a *rewriting relation* $\to_R$. This is introduced as follows: for any $s, t \in T_\Sigma$, we have $s \to_R t$ if and only if there exists a context $u \in C_\Sigma(X_1)$ and a rule $l \to r$ in R such that $s = u[l]$ and $t = u[r]$. Here we say that R rewrites s to t applying the rule $l \to r$. It is well known that the relation $\leftrightarrow_R^*$ is a congruence on the term algebra **TA**. We call $\leftrightarrow_R^*$ the congruence induced by R. A GTRS R is *equivalent to* a GTRS S, if $\leftrightarrow_R^* = \leftrightarrow_S^*$ holds.

Definition 2.4. Let R be a GTRS. Let

$$lhs(R) = \{\, t \in T_\Sigma \mid t \text{ is the left-hand side of some rule } t \to v \text{ in } R \,\}$$

be the set of left-hand sides of the rules in R, and

$$rhs(R) = \{\, t \in T_\Sigma \mid t \text{ is the right-hand side of some rule } u \to t \text{ in } R \,\}$$

be the set of right-hand sides of the rules in R. Let

$$sbt(R) = sbt(lhs(R) \cup rhs(R))$$

be the set of subterms occurring in R.

Let R be a GTRS. A ground term $t \in T_\Sigma$ is *irreducible* for R if there exists no t' such that $t \to_R t'$. The set of irreducible ground terms for R is denoted by $IRR(R)$.

- A GTRS R is *noetherian* if there exists no infinite sequence of terms

  $t_1, t_2, t_3, \ldots$ in T_Σ such that $t_1 \to_R t_2 \to_R t_3 \to_R \ldots$.

- A GTRS R is *confluent* if for any terms t_1, t_2, t_3 in T_Σ, whenever $t_1 \to_R^* t_2$ and $t_1 \to_R^* t_3$, there exists a term t_4 in T_Σ such that $t_2 \to_R^* t_4$ and $t_3 \to_R^* t_4$.

- A GTRS R is *convergent* if it is noetherian and confluent.

Let R be a convergent GTRS. It is well known that for any class Z of $\leftrightarrow_R^*$, Z contains exactly one term t in $IRR(R)$, and that for any term p in the class Z, $p \to_R^* t$. We call t the R-normal form of p and also the R-normal form of the class Z. For any term $u \in T_\Sigma$, one can effectively compute the R-normal form of u. We give a class Z of $\leftrightarrow_R^*$ through its R-normal form.

Definition 2.5. A GTRS R is *reduced* if for every rule $l \to r$ in R, l is irreducible with respect to $R - \{\, l \to r \,\}$ and r is irreducible for R.

By Definition 2.5, we get the following.

Lemma 2.6. *For any reduced GTRS R, $sbt(R) - lhs(R) \subseteq IRR(R)$ and $lhs(R) \cap rhs(R) = \emptyset$.*

We recall the following important results from [18].

Proposition 2.7. [18] *Any reduced GTRS R is convergent.*

Proposition 2.8. [18] *For a GTRS R one can effectively construct an equivalent reduced GTRS R'.*

Proposition 2.9. [18] *Let R and S be equivalent reduced GTRSs. Then $card(R) = card(S)$.*

Proposition 2.10. [18] *For a GTRS R consisting of n rules, there are at most 2^n reduced GTRSs equivalent to R.*

Consider Snyder's [18] example. Let Σ be a ranked alphabet such that $\Sigma_0 = \{a_1, b_1, a_2, b_2, \ldots, a_n, b_n\}$. Let GTRS R over Σ consist of the rules $a_1 \to b_1$, $a_2 \to b_2$, ..., $a_n \to b_n$. Then every possible reorientation of the occurrences of the arrow $\to$ yields a reduced GTRS equivalent to R. So there are 2^n reduced GTRSs equivalent to R.

Bottom-Up Tree Automata. A *bottom-up tree automaton* (bta for short) over a ranked alphabet Σ is a quadruple $\mathcal{A} = (A, \Sigma, A', R)$, where A is the finite set of states of rank 0, $\Sigma \cap A = \emptyset$, $A'(\subseteq A)$ is the set of final states, and R is the finite set of rules $f(a_1, \ldots, a_n) \to a$ with $n \geq 0$, $f \in \Sigma_n$, $a_1, \ldots, a_n, a \in A$.

We consider R as a GTRS over $\Sigma \cup A$. The tree language recognized by a bta $\mathcal{A}$ is $L(\mathcal{A}) = \{t \in T_\Sigma \mid (\exists a \in A')\ t \to_R^* a\}$. A tree language L is *recognizable* if there exists a bta $\mathcal{A}$ such that $L(\mathcal{A}) = L$ (see [13]). We give a recognizable tree language L through a bta $\mathcal{A}$ with $L = L(\mathcal{A})$.

Bta $\mathcal{A} = (A, \Sigma, A', R)$ is deterministic if for any $f \in \Sigma_n$, $n \geq 0$, $a_1, \ldots, a_n \in A$, there is at most one rule with left-hand side $f(a_1, \ldots, a_n)$ in R.

Proposition 2.11. [13] *For any btas $\mathcal{A}$ and $\mathcal{B}$, we can decide whether $L(\mathcal{A}) \subseteq L(\mathcal{B})$ and whether $L(\mathcal{A}) = L(\mathcal{B})$ and whether $L(\mathcal{A}) \cap L(\mathcal{B}) = \emptyset$.*

Proposition 2.12. [13] *For any btas $\mathcal{A}$ and $\mathcal{B}$, one can effectively construct a bta C such that $L(\mathcal{A}) \cup L(\mathcal{B}) = L(C)$.*

Proposition 2.13. [19] *For any GTRS R and any term $t \in T_\Sigma$, we can construct a bta $\mathcal{A}$ such that $L(\mathcal{A}) = [t]_{\leftrightarrow_R^*}$.*

Propositions 2.11 and 2.13 imply the following.

Proposition 2.14. *For any GTRS R and any terms $s, t \in T_\Sigma$, we can decide whether $[s]_{\leftrightarrow_R^*} = [t]_{\leftrightarrow_R^*}$.*

Note that Proposition 2.14 says that for any GTRS R and any terms $s, t \in T_\Sigma$, we can decide whether $s \leftrightarrow_R^* t$. Propositions 2.7 and 2.8 also imply Proposition 2.14.

3 Congruences and GTRSs

We adopt some basic algebraic constructs associated with GTRSs and some results on them from [10], [18], and [21]. Then we continue studying these concepts. First we introduce the concept of a set of representatives for a congruence ρ and a set of ρ-classes.

Definition 3.1. [10] Let ρ be a congruence on **TA** and let A be a set of ρ-classes. A set REP of trees is called a set of *representatives* for A if

- $REP \subseteq \bigcup A$,
- REP is closed under subtrees, and
- each class $Z \in A$ contains exactly one tree $t \in REP$.

We adopt from [10] the concept of a GTRS determined by a congruence ρ, a finite set A of ρ-classes, and a set of representatives for A.

Definition 3.2. [10] Let ρ be a congruence on **TA**, A be a finite set of ρ-classes, and REP be a set of representatives for A. Then ρ, A, and REP determine a GTRS R as follows. The rewrite rule $p \to q$ is in R if

- $p = f(p_1, \ldots, p_m)$ for some $m \geq 0$, $f \in \Sigma_m$, and $p_1, \ldots, p_m \in REP$,
- $q \in REP$,
- $p \neq q$ and $p \rho q$.

Theorem 3.14 in [10] implies the following result.

Proposition 3.3. *For any GTRS R, the set $stub(\leftrightarrow_R^*)$ is finite.*

Proposition 3.4. [18] *Let R be a GTRS, and REP be a set of representatives for $stub(\leftrightarrow_R^*)$. Then the GTRS V determined by $\leftrightarrow_R^*$, $stub(\leftrightarrow_R^*)$, and REP is reduced and is equivalent to R.*

Proof. By Proposition 3.3, $stub(\leftrightarrow_R^*)$ is finite. By Lemma 3.7 in [10], V is reduced. By Lemma 3.10 in [10], V is equivalent to R. □

We now adopt Lemma 3.13 in [10].

Proposition 3.5. [10]. *For any GTRS R,*

$$trunk(\underset{R}{\overset{*}{\leftrightarrow}}) \subseteq \bigcup [sbt(R)]_{\leftrightarrow_R^*}.$$

We now adopt Lemma 3.21 in [21].

Proposition 3.6. [21] *For any reduced GTRS R,*

$$trunk(\overset{*}{\underset{R}{\leftrightarrow}}) = \bigcup [sbt(R)]_{\leftrightarrow^*_R} .$$

Proposition 3.6 says that $trunk(\leftrightarrow^*_R)$ is the union of finitely many $\leftrightarrow^*_R$ classes. By Propositions 2.12, 2.13, and 3.6 we have the following.

Proposition 3.7. *For any reduced GTRS R, we can effectively construct a bta $\mathcal{A}$ such that $L(\mathcal{A}) = trunk(\leftrightarrow^*_R)$.*

Proposition 3.8. *For any reduced GTRS R,*

$$stub(\overset{*}{\underset{R}{\leftrightarrow}}) = [sbt(R)]_{\leftrightarrow^*_R} .$$

Proof. Proposition 3.6 implies our assertion. □

Proposition 3.9. *For any GTRS R, we can construct $stub(\leftrightarrow^*_R)$.*

Proof. By Propositions 2.14, and 3.8, we construct $stub(\leftrightarrow^*_R)$. □
By Definition 3.2 and Propositions 2.14 and 3.9 we have the following.

Proposition 3.10. *Let S be a GTRS. Let REP be a set of representatives for $stub(\leftrightarrow^*_S)$. Let $\leftrightarrow^*_S$, $stub(\leftrightarrow^*_S)$, and REP determine the reduced GTRS R. Then we construct the reduced GTRS R.*

In the proofs of Theorem 3.18 and Theorem 4.6 in [18], Snyder showed the following important result.

Proposition 3.11. [18] *Let R be a reduced GTRS, and let R′ be an arbitrary reduced GTRS equivalent to R. Then we can effectively construct a set REP of representatives for $[sbt(R)]_{\leftrightarrow^*_R}$ such that the GTRS determined by $\leftrightarrow^*_R$, $[sbt(R)]_{\leftrightarrow^*_R}$, and REP is equal to R′.*

The following result is an important consequence of Proposition 3.11.

Proposition 3.12. [18] *Let R be a reduced GTRS. Then we can effectively construct a set REP of representatives for $[sbt(R)]_{\leftrightarrow^*_R}$ such that the GTRS determined by $\leftrightarrow^*_R$, $[sbt(R)]_{\leftrightarrow^*_R}$, and REP is equal to R.*

Lemma 3.13. *Let R be a reduced GTRS, and let REP be a set of representatives for $[sbt(R)]_{\leftrightarrow^*_R}$ such that the GTRS determined by $\leftrightarrow^*_R$, $[sbt(R)]_{\leftrightarrow^*_R}$, and REP is equal to R. Then $REP = sbt(R) - lhs(R)$.*

Proof. By Definitions 3.1 and 3.2, we have

$$sbt(R) - lhs(R) \subseteq REP$$

and

$$REP \subseteq IRR(R)\,. \tag{3}$$

We now show that

$$REP \subseteq sbt(R) - lhs(R)\,.$$

Let $s \in REP$ be arbitrary. Since REP is a set of representatives for $[sbt(R)]_{\leftrightarrow_R^*}$, there is $t \in sbt(R)$ such that $s \leftrightarrow_R^* t$. If $t \in sbt(R) - lhs(R)$, then $t \in IRR(R)$, see Lemma 2.6. If $t \in lhs(R)$, then there is a rule $t \to r$ in R. By Definition 2.5, $r \in sbt(R) - lhs(R)$ and $r \in IRR(R)$. Thus, there is $u \in sbt(R) - lhs(R)$ such that $u \in IRR(R)$ and

$$s \overset{*}{\underset{R}{\leftrightarrow}} u\,. \tag{4}$$

By Proposition 2.7, GTRS R is convergent. By (3), $s, u \in IRR(R)$. By (4), we get that $s = u$. Thus $REP \subseteq sbt(R) - lhs(R)$. □

Theorem 3.14. *For any reduced GTRS R,*
(a) $sbt(R) - lhs(R)$ *is a set of representatives for* $[sbt(R)]_{\leftrightarrow_R^*}$*, and*
(b) the GTRS determined by $\leftrightarrow_R^*$, $[sbt(R)]_{\leftrightarrow_R^*}$*, and* $sbt(R) - lhs(R)$ *is equal to R.*

Proof. By Lemma 2.6,

$$sbt(R) - lhs(R) \subseteq IRR(R)\,. \tag{5}$$

Apparently, $sbt(R) - lhs(R) \subseteq sbt(R) \subseteq \bigcup [sbt(R)]_{\leftrightarrow_R^*}$. By Definition 2.5, $sbt(R) - lhs(R)$ is closed under subtrees. Since R is convergent, by (5), each class $Z \in [sbt(R)]_{\leftrightarrow_R^*}$ contains exactly one tree $t \in sbt(R) - lhs(R)$. Hence $sbt(R) - lhs(R)$ is a set of representatives for $[sbt(R)]_{\leftrightarrow_R^*}$. The congruence $\leftrightarrow_R^*$, the set $[sbt(R)]_{\leftrightarrow_R^*}$, and the set of representatives $sbt(R) - lhs(R)$ for $[sbt(R)]_{\leftrightarrow_R^*}$ determine a GTRS S. We now show that $R = S$.

First we show that $R \subseteq S$. Take an arbitrary rewrite rule $p \to q$ in R. Then

- $p = f(p_1, \ldots, p_m)$ for some $m \geq 0$, $f \in \Sigma_m$, and $p_1, \ldots, p_m \in sbt(R) - lhs(R)$,
- $q \in sbt(R) - lhs(R)$,
- $p \neq q$ and $p \leftrightarrow_R^* q$.

By the definition of S, the rewrite rule $p \to q$ is in S.

Second we show that $S \subseteq R$. Take an arbitrary rewrite rule $p \to q$ in S. Then

- $p = f(p_1, \ldots, p_m)$ for some $m \geq 0$, $f \in \Sigma_m$, and $p_1, \ldots, p_m \in sbt(R) - lhs(R)$,

- $q \in sbt(R) - lhs(R)$,
- $p \neq q$ and $p \leftrightarrow_R^* q$.

By Lemma 2.6,

$$p_1, \ldots, p_m, q \in IRR(R)\,. \tag{6}$$

Since R is convergent, $p \rightarrow_R^* q$. Thus there is a rule $p \rightarrow w$ in R. As R is a reduced GTRS, we have

$$w \in IRR(R)\,. \tag{7}$$

Furthermore,

$$w \overset{*}{\underset{R}{\leftrightarrow}} q\,. \tag{8}$$

By (6),(7), and (8), $w = q$. Therefore, $p \rightarrow q$ is in R. □

We note that Proposition 3.12 and Lemma 3.13 also imply Theorem 3.14. By Proposition 3.8 and Theorem 3.14 we have the following.

Corollary 3.15. *For any reduced GTRS R,*
(a) $sbt(R) - lhs(R)$ is a set of representatives for $stub(\leftrightarrow_R^*)$, and
(b) the GTRS determined by $\leftrightarrow_R^*$, $stub(\leftrightarrow_R^*)$, and $sbt(R) - lhs(R)$ is equal to R.

Definition 3.16. *Let R be a reduced GTRS, and let Z be a compound class of $\leftrightarrow_R^*$. The compound degree $deg(Z)$ of Z is the number of all compound equalities $f^{\mathbf{TA}/\leftrightarrow_R^*}(Z_1, \ldots, Z_m) = Z$.*

4 Stub equalities and compound equalities

In this section we study the stub equalities and compound equalities of GTRSs.

Lemma 4.1. *Let R be a reduced GTRS. Then there is a bijective mapping ψ : $STY \rightarrow sbt(R)$.*

Proof. By Corollary 3.15,
(a) $sbt(R) - lhs(R)$ is a set of representatives for $stub(\leftrightarrow_R^*)$, and
(b) the GTRS determined by $\leftrightarrow_R^*$, $stub(\leftrightarrow_R^*)$, and $sbt(R) - lhs(R)$ is equal to R.
We define a mapping $\psi : STY \rightarrow sbt(R)$ as follows. Consider a stub equality

$$f^{\mathbf{TA}/\leftrightarrow_R^*}(Z_1, \ldots, Z_m) = Z\,. \tag{9}$$

in STY. By Lemma 2.2, $Z_1, \ldots, Z_m \in stub(\leftrightarrow_R^*)$. There are $t_1, \ldots, t_m, t \in sbt(R) - lhs(R)$ such that $[t_i]_{\leftrightarrow_R^*} = Z_i$ for $1 \leq i \leq m$ and $[t]_{\leftrightarrow_R^*} = Z$. Then we assign $f(t_1, \ldots, t_m)$ to the compound equality (9). We now show that $f(t_1, \ldots, t_m) \in sbt(R)$. We distinguish two cases.

Case 1: $f(t_1, \ldots, t_m) \in sbt(R) - lhs(R)$. By (9), $[f(t_1, \ldots, t_m)]_{\leftrightarrow^*_R} = Z$. Consequently, $f(t_1, \ldots, t_m) = t$. We assign $t \in sbt(R) - lhs(R)$ to the stub equality (9).

Case 2: $f(t_1, \ldots, t_m) \in lhs(R)$. Then by Definition 3.2 and Condition (b), the ground term rewrite rule $f(t_1, \ldots, t_m) \to t$ is in R. We assign $f(t_1, \ldots, t_m) \in lhs(R)$ to the stub equality (9).

We now show that ψ is injective. Assume that ψ assigns $t \in sbt(R)$ to the stub equalities (9) and

$$g^{\mathbf{TA}/\leftrightarrow^*_R}(W_1, \ldots, W_n) = W\,. \tag{10}$$

Then $t \in Z$ and $t \in W$. Hence $Z = W$. Let $t_1, \ldots, t_m \in sbt(R) - lhs(R)$ be such that $[t_i]_{\leftrightarrow^*_R} = Z_i$ for $1 \leq i \leq m$. Let $s_1, \ldots, s_n \in sbt(R) - lhs(R)$ be such that $[s_i]_{\leftrightarrow^*_R} = W_i$ for $1 \leq i \leq n$. By the definition of ψ, $t = f(t_1, \ldots, t_m) = g(s_1, \ldots, s_n)$. Consequently, $f = g$, $m = n$, and $t_i = s_i$ for $1 \leq i \leq m$. By the definition of t_i and s_i, $Z_i = W_i$ for $1 \leq i \leq m$. Thus (9) is equal to (10).

We now show that ψ is surjective. First, consider an arbitrary element of $lhs(R)$. Then it is of the form $f(t_1, \ldots, t_m)$, where $f \in \Sigma_m$, $m \geq 0$, $t_1, \ldots, t_m \in sbt(R) - lhs(R)$. There is a ground term rewrite rule $f(t_1, \ldots, t_m) \to t$ in R. Then the stub equality

$$f^{\mathbf{TA}/\leftrightarrow^*_R}([t_1]_{\leftrightarrow^*_R}, \ldots, [t_m]_{\leftrightarrow^*_R}) = [t]_{\leftrightarrow^*_R} \tag{11}$$

is in STY. Mapping ψ assigns $f(t_1, \ldots, t_m)$ to the stub equality (11).

Second, consider an arbitrary element $g(s_1, \ldots, s_n)$ of $sbt(R) - lhs(R)$. Here $g \in \Sigma_n$, $n \geq 0$, $s_1, \ldots, s_n \in sbt(R) - lhs(R)$. By Proposition 3.8, $[g(s_1, \ldots, s_n)]_{\leftrightarrow^*_R} \in stub(\leftrightarrow^*_R)$. Then the stub equality

$$g^{\mathbf{TA}/\leftrightarrow^*_R}([s_1]_{\leftrightarrow^*_R}, \ldots, [s_n]_{\leftrightarrow^*_R}) = [g(s_1, \ldots, s_n)]_{\leftrightarrow^*_R} \tag{12}$$

is in STY, where $g \in \Sigma_n$, $n \geq 0$. Mapping ψ assigns $g(s_1, \ldots, s_n)$ to the stub equality (12). □

Lemma 4.2. *Let R be a reduced GTRS. Then (a)-(d) hold:*

(a) There is a bijective mapping $\phi : COY \to lhs(R) \cup rhs(R)$.

(b) There is a bijective mapping $\xi : COY \to R \cup rhs(R)$.

(c) For each compound class $[t]_{\leftrightarrow^*_R}$ with $t \in rhs(R)$, there are $deg([t]_{\leftrightarrow^*_R}) - 1$ rules with right-hand side t in R.

(d) $card(COY) = card(lhs(R)) + card(rhs(R))$.

Proof. By Corollary 3.15,

(i) $sbt(R) - lhs(R)$ is a set of representatives for $stub(\leftrightarrow^*_R)$, and

(ii) the GTRS determined by $\leftrightarrow^*_R$, $stub(\leftrightarrow^*_R)$, and $sbt(R) - lhs(R)$ is equal to R.

First we show (a). We define a mapping $\phi : COY \to lhs(R) \cup rhs(R)$ as follows. Consider a compound equality

$$f^{\mathbf{TA}/\leftrightarrow^*_R}(Z_1, \ldots, Z_m) = Z\,. \tag{13}$$

in COY. By Lemma 2.2, $Z_1, \ldots, Z_m \in stub(\leftrightarrow_R^*)$. Let $t_1, \ldots, t_m, t \in sbt(R) - lhs(R)$ be such that $[t_i]_{\leftrightarrow_R^*} = Z_i$ for $1 \le i \le m$ and $[t]_{\leftrightarrow_R^*} = Z$. Then we assign $f(t_1, \ldots, t_m)$ to the compound equality (13). We now show that $f(t_1, \ldots, t_m) \in lhs(R) \cup rhs(R)$. We distinguish two cases.

Case 1: $f(t_1, \ldots, t_m) \in sbt(R) - lhs(R)$. By (13), $[f(t_1, \ldots, t_m)]_{\leftrightarrow_R^*} = Z$. Consequently, $f(t_1, \ldots, t_m) = t$. By (i), (ii), and Definition 3.2, $t \in rhs(R)$. We assign $t \in rhs(R)$ to the compound equality (13).

Case 2: $f(t_1, \ldots, t_m) \in lhs(R)$. Then by Definition 3.2 and Condition (b), the ground term rewrite rule $f(t_1, \ldots, t_m) \to t$ is in R. Then we assign $f(t_1, \ldots, t_m) \in lhs(R)$ to the compound equality (13).

Observe that mapping ψ, defined in the proof of Lemma 4.1, is an extension of the mapping ϕ.

We now show that ϕ is injective. Assume that ϕ assigns $t \in lhs(R) \cup rhs(R)$ to the compound equalities (13) and

$$g^{\mathbf{TA}/\leftrightarrow_R^*}(W_1, \ldots, W_n) = W, \tag{14}$$

where $g \in \Sigma_n$, $n \ge 0$. Then $t \in Z$ and $t \in W$. Hence $Z = W$. Let $t_1, \ldots, t_m \in sbt(R) - lhs(R)$ be such that $[t_i]_{\leftrightarrow_R^*} = Z_i$ for $1 \le i \le m$. Let $s_1, \ldots, s_n \in sbt(R) - lhs(R)$ be such that $[s_i]_{\leftrightarrow_R^*} = W_i$ for $1 \le i \le n$.

By the definition of ϕ, $t = f(t_1, \ldots, t_m) = g(s_1, \ldots, s_n)$. Consequently, $f = g$, $m = n$, and $t_i = s_i$ for $1 \le i \le m$. By the definition of t_i and s_i, $Z_i = W_i$ for $1 \le i \le m$. Therefore (13) is equal to (14).

We now show that ϕ is surjective. First, consider an arbitrary element of $lhs(R)$. Then it is of the form $f(t_1, \ldots, t_m)$, where $f \in \Sigma_m$, $m \ge 0$, $t_1, \ldots, t_m \in T_\Sigma$. There is a ground term rewrite rule $f(t_1, \ldots, t_m) \to t$ in R. Then the compound equality

$$f^{\mathbf{TA}/\leftrightarrow_R^*}([t_1]_{\leftrightarrow_R^*}, \ldots, [t_m]_{\leftrightarrow_R^*}) = [t]_{\leftrightarrow_R^*} \tag{15}$$

is in COY. Mapping ϕ assigns $f(t_1, \ldots, t_m)$ to the compound equality (15).

Second, consider an arbitrary element $g(s_1, \ldots, s_n)$ of $rhs(R)$ with $g \in \Sigma_n$, $n \ge 0$, $s_1, \ldots, s_n \in sbt(R) - lhs(R)$. Then the compound equality

$$g^{\mathbf{TA}/\leftrightarrow_R^*}([s_1]_{\leftrightarrow_R^*}, \ldots, [s_n]_{\leftrightarrow_R^*}) = [g(s_1, \ldots, s_n)]_{\leftrightarrow_R^*} \tag{16}$$

is in COY. Mapping ϕ assigns $g(s_1, \ldots, s_n)$ to the compound equality (16). The proof of (a) is complete.

Condition (a) implies Condition (b) and Condition (d). Definition 3.16 and the construction of ϕ in the proof of (a) shows Condition (c). □

Theorem 4.3. *For a given reduced GTRS R, we can effectively construct the sets COY and STY.*

Proof. By Lemma 4.1, and the definition of the mapping ϕ in the proof of Lemma 4.1, STY consists of all stub equalities

$$f^{\mathbf{TA}/\leftrightarrow^*_R}([t_1]_{\leftrightarrow^*_R}, \ldots, [t_m]_{\leftrightarrow^*_R}) = [f(t_1, \ldots, t_m)]_{\leftrightarrow^*_R},$$

where $f(t_1, \ldots, t_m) \in sbt(R)$. By (a), Lemma 4.2, and the definition of the mapping ϕ in the proof of (a), Lemma 4.2, COY consists of all compound equalities

$$f^{\mathbf{TA}/\leftrightarrow^*_R}([t_1]_{\leftrightarrow^*_R}, \ldots, [t_m]_{\leftrightarrow^*_R}) = [f(t_1, \ldots, t_m)]_{\leftrightarrow^*_R},$$

where $f(t_1, \ldots, t_m) \in lhs(R) \cup rhs(R)$. By Proposition 2.13, the proof is complete. $\square$

Proposition 2.8 and Theorem 4.3 imply the following result.

Consequence 4.4. *For a given GTRS R, we can effectively construct STY and COY.*

We now discuss the connections of Consequence 4.4 with the results of Fülöp and Vágvölgyi [9]. They [9] introduced the following concepts and showed the following results. Let E be a GTRS over a ranked alphabet Σ, and let

$$\Theta = \overset{*}{\underset{E}{\leftrightarrow}} \cap (sbt(E) \times sbt(E)) .$$

Then Θ is an equivalence relation on $sbt(E)$. Furthermore, for any $t \in sbt(E)$, we have $[t]_\Theta = [t]_{\leftrightarrow^*_E} \cap sbt(E)$. Let $CLS = \{ [t]_\Theta \mid t \in sbt(E) \}$. We can effectively construct Θ and CLS. Consider the ranked alphabet $\Sigma \cup CLS$, where the elements of CLS are viewed as symbols with rank 0. We now define the GTRS R over $\Sigma \cup CLS$. GTRS R consists of all rules

$$f([t_1]_\Theta, \ldots, [t_m]_\Theta) \to [f(t_1, \ldots, t_m)]_\Theta \tag{17}$$

where $f \in \Sigma_m$, $m \geq 0$, $t_1, \ldots, t_m, f(t_1, \ldots, t_m) \in sbt(E)$, and $[t_1]_\Theta, \ldots, [t_m]_\Theta \in CLS$, and $[f(t_1, \ldots, t_m)]_\Theta \in CLS$. GTRS R is reduced, and $\leftrightarrow^*_E = \leftrightarrow^*_R \cap T_\Sigma \times T_\Sigma$. For every $t \in sbt(E)$, we have $t \to^*_R [t]_\Theta$. Furthermore, one can effectively construct the GTRS R. On the basis of the above concepts and results of Fülöp and Vágvölgyi [9], we define the mapping $\phi : STY(\leftrightarrow^*_E) \to R$ as follows. To each $stub(\leftrightarrow^*_E)$ equality

$$f^{\mathbf{TA}/\leftrightarrow^*_E}([t_1]_{\leftrightarrow^*_E}, \ldots, [t_m]_{\leftrightarrow^*_E}) = [f(t_1, \ldots, t_m)]_{\leftrightarrow^*_E} \tag{18}$$

with $t_1, \ldots, t_m, f(t_1, \ldots, t_m) \in sbt(E)$, ϕ assigns the rule (17) of R. Apparently, ϕ is an injective mapping. Observe that $\mathcal{A} = (CLS, \Sigma, \emptyset, R)$ is a deterministic bta.

Using ϕ, we construct the set CP of all states $[t]_\Theta \in CLS$, where $[t]_{\leftrightarrow^*_E}$ is a compound $\leftrightarrow^*_E$ class. It is well known that for any given state $[t]_\Theta \in CP$, we can construct the set of all states $[p]_\Theta \in CLS$, such that $u[[p]_\Theta] \to_R^* [t]_\Theta$ for some context $u \in C_\Sigma(X_1)$. Therefore, we construct the set $CLS1$ of all states $[t]_\Theta \in CLS$, where $[t]_{\leftrightarrow_E^*}$ is in $stub(\leftrightarrow_E^*)$. Then we define the GTRS S from R by dropping all rules (17) of R such that $[f(t_1, \dots, t_m)]_\Theta$ is not in $CLS1$. One can effectively construct the GTRS S. Observe that the range of ϕ is equal to S. Hence we can write ϕ in the following form: $\phi : STY(\leftrightarrow_E^*) \to S$. Here ϕ is a bijective mapping. For each $[t]_\Theta \in CLS$, define the deterministic bta $\mathcal{A}\langle [t]_\Theta \rangle = (CLS, \Sigma, \{ [t]_\Theta \}, R)$. Then $L(\mathcal{A}\langle [t]_\Theta \rangle) = [t]_{\leftrightarrow_R^*}$. Consequently, we can give (18) by (17).

Fülöp and Vágvölgyi [11] constructed a reduced GTRS Q over Σ such that $\leftrightarrow_E^* = \leftrightarrow_R^* \cap T_\Sigma \times T_\Sigma = \leftrightarrow_Q^*$. They [11] observed that they obtained a new ground completion algorithm which works as follows. Given a GTRS E, we construct a reduced GTRS equivalent to E in two steps. In the first step, we compute the reduced GTRS R over $\Sigma \cup CLS$. Then in the second step, we construct the reduced GTRS Q over Σ. This ground completion parallels to Snyder's fast algorithm, see [18], and the results of [14] and [17].

5 Union of GTRSs

We study the congruence generated by the union $R \cup S$ of GTRSs R and S, where the congruences generated by R and S intersect with respect to their stubs. Then we study the congruence $\leftrightarrow_{R1\cup\dots\cup Rn}^*$, where $R1, R2, \dots, Rn$, $n \geq 2$, are GTRSs and any two of $\leftrightarrow_{R1}^*, \dots, \leftrightarrow_{Rn}^*$ intersect with respect to their stubs.

Lemma 5.1. *Let R and S be reduced GTRSs such that $\leftrightarrow_R^*$ and $\leftrightarrow_S^*$ intersect with respect to their stubs. Then Conditions (i)-(vi) hold.*

(i) For any $p \in trunk(\leftrightarrow_R^*)$ and $t \in T_\Sigma$, if $p \leftrightarrow_{R\cup S}^* t$, then $p \leftrightarrow_R^* t$.

(ii) For any $p \in trunk(\leftrightarrow_R^*)$, $[p]_{\leftrightarrow_R^*} = [p]_{\leftrightarrow_{R\cup S}^*}$.

(iii) For any $p \in sbt(R)$, $[p]_{\leftrightarrow_R^*} = [p]_{\leftrightarrow_{R\cup S}^*}$.

(iv) $comp(\leftrightarrow_R^*) \cup comp(\leftrightarrow_S^*) \subseteq comp(\leftrightarrow_{R\cup S}^*)$.

(v) $trunk(\leftrightarrow_R^*) \cup trunk(\leftrightarrow_S^*) \subseteq trunk(\leftrightarrow_{R\cup S}^*)$.

(vi) $stub(\leftrightarrow_R^*) \cup stub(\leftrightarrow_S^*) \subseteq stub(\leftrightarrow_{R\cup S}^*)$.

Proof. First we show the following.

(a) For any $p \in trunk(\leftrightarrow_R^*)$ and $t \in T_\Sigma$, if $p \leftrightarrow_S t$, then $p \leftrightarrow_R^* t$.

Let $p \in trunk(\leftrightarrow_R^*)$ be arbitray. First, assume $p \to_S t$ for some $t \in T_\Sigma$. Then there is a rule $l \to r$ of S and a context $u \in C_\Sigma(X_1)$ such that $p = u[l]$ and $t = u[r]$. Since $p \in trunk(\leftrightarrow_R^*)$, by Proposition 2.1, $l \in trunk(\leftrightarrow_R^*)$. Hence $[l]_{\leftrightarrow_R^*} \in stub(\leftrightarrow_R^*)$. By Proposition 3.8, $[l]_{\leftrightarrow_S^*} \in stub(\leftrightarrow_S^*)$. Observe that $l \in [l]_{\leftrightarrow_R^*}$ and $l \in [l]_{\leftrightarrow_S^*}$. By the assumption of the lemma, $[l]_{\leftrightarrow_R^*} = [l]_{\leftrightarrow_S^*}$. Hence $l \leftrightarrow_R^* r$. Therefore $p \leftrightarrow_R^* t$.

Second, assume $t \to_S p$ for some $t \in T_\Sigma$. Then there is a rule $l \to r$ of S and a context $u \in C_\Sigma(X_1)$ such that $t = u[l]$ and $p = u[r]$. Since $p \in trunk(\leftrightarrow_R^*)$, by

Proposition 2.1, $r \in trunk(\leftrightarrow_R^*)$. Hence $[r]_{\leftrightarrow_R^*} \in stub(\leftrightarrow_R^*)$. By Proposition 3.8, $[r]_{\leftrightarrow_S^*} \in stub(\leftrightarrow_S^*)$. Observe that $r \in [r]_{\leftrightarrow_R^*}$ and $r \in [r]_{\leftrightarrow_S^*}$. By the assumption of the lemma, $[r]_{\leftrightarrow_R^*} = [r]_{\leftrightarrow_S^*}$. Hence $l \leftrightarrow_R^* r$. Consequently $p \leftrightarrow_R^* t$.

By Proposition 3.6, Condition (a) implies (b).

(b) For any $p \in trunk(\leftrightarrow_R^*)$ and $t \in T_\Sigma$, if $p \leftrightarrow_S^* t$, then $p \leftrightarrow_R^* t$.

Condition (b) implies Condition (i) of the lemma. Condition (i) implies Condition (ii). Proposition 3.6 and Condition (ii) imply Condition (iii) of the lemma.

We now show (iv). First we show that

$$comp(\underset{R}{\overset{*}{\leftrightarrow}}) \subseteq comp(\underset{R\cup S}{\overset{*}{\leftrightarrow}}) . \tag{19}$$

Consider an arbitrary compound $\leftrightarrow_R^*$ class Z. Then there are $comp(\leftrightarrow_R^*)$ equalities

$$f^{\mathbf{TA}/\leftrightarrow_R^*}(Z_1, \ldots, Z_m) = Z , \tag{20}$$

and

$$g^{\mathbf{TA}/\leftrightarrow_R^*}(W_1, \ldots, W_n) = Z , \tag{21}$$

where $Z_1, \ldots, Z_m, W_1, \ldots, W_n$ are $\leftrightarrow_R^*$-classes, and $f \neq g$ or ($f = g$ and there is $j \in \{1, \ldots, , n\}$ such that $z_j \neq w_j$). By Lemma 2.2, $Z, Z_1, \ldots, Z_m, W_1, \ldots, W_n$ are in $stub(\leftrightarrow_R^*)$. By (ii) $Z, Z_1, \ldots, Z_m, W_1, \ldots, W_n$ are $\leftrightarrow_{R\cup S}^*$-classes as well. Thus we get the $stub(\leftrightarrow_{R\cup S}^*)$ equalities

$$f^{\mathbf{TA}/\leftrightarrow_{R\cup S}^*}(Z_1, \ldots, Z_m) = Z , \tag{22}$$

and

$$g^{\mathbf{TA}/\leftrightarrow_{R\cup S}^*}(W_1, \ldots, W_n) = Z . \tag{23}$$

Consequently, Z is a $comp(\leftrightarrow_{R\cup S}^*)$ class. Hence (19) holds.

The proof of

$$comp(\underset{S}{\overset{*}{\leftrightarrow}}) \subseteq comp(\underset{R\cup S}{\overset{*}{\leftrightarrow}}) . \tag{24}$$

is symmetrical to that of (19). By (19) and (24), we have (iv).

Condition (iv) implies Condition (v). Conditions (ii) and (v) imply Condition (vi). □

Theorem 5.2. *For any GTRSs R and S, the following two conditions are equivalent.*

(i) $\leftrightarrow_R^*$ and $\leftrightarrow_S^*$ intersect with respect to their stubs.

(ii) $stub(\leftrightarrow_{R\cup S}^*) = stub(\leftrightarrow_R^*) \cup stub(\leftrightarrow_S^*)$.

Proof. By Proposition 2.8, we may assume that GTRSs R and S are reduced. Assume that (i) holds. By (vi), Lemma 5.1, we have

$$stub(\underset{R}{\overset{*}{\leftrightarrow}}) \cup stub(\underset{S}{\overset{*}{\leftrightarrow}}) \subseteq stub(\underset{R\cup S}{\overset{*}{\leftrightarrow}}) . \tag{25}$$

We now show that

$$stub(\overset{*}{\underset{R\cup S}{\leftrightarrow}}) \subseteq stub(\overset{*}{\underset{R}{\leftrightarrow}}) \cup stub(\overset{*}{\underset{S}{\leftrightarrow}})\,. \tag{26}$$

$stub(\leftrightarrow^*_{R\cup S}) \subseteq \{\, [t]_{\leftrightarrow^*_{R\cup S}} \mid t \in sbt(R) \cup sbt(S)\,\} =$ (by Proposition 3.5)
$\{\, [t]_{\leftrightarrow^*_R} \mid t \in sbt(R)\,\} \cup \{\, [t]_{\leftrightarrow^*_S} \mid t \in sbt(S)\,\} =$ (by (iii), Lemma 5.1)
$stub(\leftrightarrow^*_R) \cup stub(\leftrightarrow^*_S)$ (by Proposition 3.8).
Thus (26) holds. (25) and (26) imply (ii).

Assume that (ii) holds. Let $Z_1 \in stub(\leftrightarrow^*_R)$ and $Z_2 \in stub(\leftrightarrow^*_S)$ be arbitary. Then $Z_1, Z_2 \in stub(\leftrightarrow^*_{R\cup S})$. Hence $Z_1 \cap Z_2 = \emptyset$ or $Z_1 = Z_2$. □

Lemma 5.3. *For any GTRSs R and S, we can decide whether $\leftrightarrow^*_R$ and $\leftrightarrow^*_S$ intersect with respect to their stubs.*

Proof. By Proposition 3.9, we construct $stub(\leftrightarrow^*_R)$ and $stub(\leftrightarrow^*_S)$. Then for all $Z_1 \in stub(\leftrightarrow^*_R)$ and $Z_2 \in stub(\leftrightarrow^*_S)$, we decide whether $Z_1 \cap Z_2 \neq \emptyset$ and whether $Z_1 = Z_2$, see Proposition 2.11. $\leftrightarrow^*_R$ and $\leftrightarrow^*_S$ intersect with respect to their stubs if and only if for any $Z_1 \in \leftrightarrow^*_R$ and $Z_2 \in \leftrightarrow^*_S$, $Z_1 \cap Z_2 = \emptyset$ or $Z_1 = Z_2$. □

Theorem 5.4. *For any GTRSs R and S, the following two conditions are equivalent.*

(a) $\leftrightarrow^*_R$ and $\leftrightarrow^*_S$ intersect with respect to their stubs.
(b) $STN(\leftrightarrow^*_{R\cup S}) = STN(\leftrightarrow^*_R) \cup STN(\leftrightarrow^*_S)$.

Proof. By Proposition 2.8, we may assume that GTRSs R and S are reduced. Assume that (a) holds. By Theorem 5.2,

$$stub(\overset{*}{\underset{R}{\leftrightarrow}}) \cup stub(\overset{*}{\underset{S}{\leftrightarrow}}) = stub(\overset{*}{\underset{R\cup S}{\leftrightarrow}})\,. \tag{27}$$

We now show that

$$STN(\overset{*}{\underset{R\cup S}{\leftrightarrow}}) \subseteq STN(\overset{*}{\underset{R}{\leftrightarrow}}) \cup STN(\overset{*}{\underset{S}{\leftrightarrow}})\,. \tag{28}$$

Consider an arbitrary stub equation

$$f(Z_1, \ldots, Z_m) \approx Z \tag{29}$$

in $STN(\leftrightarrow^*_{R\cup S})$. Then

$$f^{\mathbf{TA}/\leftrightarrow^*_{R\cup S}}(Z_1, \ldots, Z_m) = Z \tag{30}$$

and $Z \in stub(\leftrightarrow^*_{R\cup S})$. By (27), $Z \in stub(\leftrightarrow^*_R)$ or $Z \in stub(\leftrightarrow^*_S)$. First assume that $Z \in stub(\leftrightarrow^*_R)$. Let $t_i \in Z_i$ for $i = 1, \ldots, m$. Then $f(t_1, \ldots, t_m) \in Z$. Consequently, $f(t_1, \ldots, t_m) \in trunk(\leftrightarrow^*_R)$. By Proposition 2.1, $t_1, \ldots, t_m \in trunk(\leftrightarrow^*_R)$. By (ii), Lemma 5.1, $[t_i]_{\leftrightarrow^*_R} = [t_i]_{\leftrightarrow^*_{R\cup S}} = Z_i$ for $i = 1, \ldots, m$. Hence by (30) we have

$$f^{\mathbf{TA}/\leftrightarrow^*_R}(Z_1, \ldots, Z_m) = Z\,.$$

Thus (29) is in $STN(\leftrightarrow_R^*)$. Second assume that $Z \in stub(\leftrightarrow_S^*)$. Symmetrically to the first case, we get that (29) is in $STN(\leftrightarrow_S^*)$. The proof of (28) is complete.

We now show that

$$STN(\underset{R}{\overset{*}{\leftrightarrow}}) \cup STN(\underset{S}{\overset{*}{\leftrightarrow}}) \subseteq STN(\underset{R\cup S}{\overset{*}{\leftrightarrow}})\,. \tag{31}$$

First we show that

$$STN(\underset{R}{\overset{*}{\leftrightarrow}}) \subseteq STN(\underset{R\cup S}{\overset{*}{\leftrightarrow}})\,. \tag{32}$$

Consider an arbitrary stub equation

$$f(Z_1,\ldots,Z_m) \approx Z \tag{33}$$

in $STN(\leftrightarrow_R^*)$. Then

$$f^{\mathbf{TA}/\leftrightarrow_R^*}(Z_1,\ldots,Z_m) = Z$$

and $Z_1,\ldots,Z_m,Z \in stub(\leftrightarrow_R^*)$. By (27), $Z_1,\ldots,Z_m,Z \in stub(\leftrightarrow_{R\cup S}^*)$. Consequently

$$f^{\mathbf{TA}/\leftrightarrow_{R\cup S}^*}(Z_1,\ldots,Z_m) = Z\,.$$

Hence (33) is in $STN(\leftrightarrow_{R\cup S}^*)$. Hence (32) holds. Symmetrically, we get that $STN(\leftrightarrow_S^*) \subseteq STN(\leftrightarrow_{R\cup S}^*)$. The proof of (31) is complete. (28) and (31) imply Condition (b).

Assume that (b) holds. We now show that

$$stub(\underset{R}{\overset{*}{\leftrightarrow}}) \cup stub(\underset{S}{\overset{*}{\leftrightarrow}}) = stub(\underset{R\cup S}{\overset{*}{\leftrightarrow}})\,. \tag{34}$$

First we show that

$$stub(\underset{R}{\overset{*}{\leftrightarrow}}) \cup stub(\underset{S}{\overset{*}{\leftrightarrow}}) \subseteq stub(\underset{R\cup S}{\overset{*}{\leftrightarrow}})\,. \tag{35}$$

We show that

$$stub(\underset{R}{\overset{*}{\leftrightarrow}}) \subseteq stub(\underset{R\cup S}{\overset{*}{\leftrightarrow}})\,. \tag{36}$$

Let $Z \in stub(\leftrightarrow_R^*)$. Then $Z = [t]_{\leftrightarrow_R^*}$ for some $t \in sbt(R)$, see Proposition 3.8. $t = f(t_1,\ldots,t_m)$ for some $f \in \Sigma_m$, $m \geq 0$, and $t_1,\ldots,t_m \in sbt(R)$. Hence the stub equation

$$f([t_1]_{\leftrightarrow_R^*},\ldots,[t_m]_{\leftrightarrow_R^*}) \approx [t]_{\leftrightarrow_R^*} \tag{37}$$

is in $STN(\leftrightarrow_R^*)$. By (b), stub equation (37) is in $STN(\leftrightarrow_{R\cup S}^*)$. Hence $[t]_{\leftrightarrow_R^*}$ is in $stub(\leftrightarrow_{R\cup S}^*)$. Thus (36) holds. One can show that

$$stub(\underset{S}{\overset{*}{\leftrightarrow}}) \subseteq stub(\underset{R\cup S}{\overset{*}{\leftrightarrow}}) \tag{38}$$

symmetrically. (36) and (38) imply (35).

Second we show that

$$stub(\underset{R\cup S}{\overset{*}{\leftrightarrow}}) \subseteq stub(\underset{R}{\overset{*}{\leftrightarrow}}) \cup stub(\underset{S}{\overset{*}{\leftrightarrow}})\,. \tag{39}$$

Let $Z \in stub(\leftrightarrow^*_{R\cup S})$. Then $Z = [t]_{\leftrightarrow^*_{R\cup S}}$ for some $t \in sbt(R \cup S)$, see Proposition 3.5. $t = f(t_1, \ldots, t_m)$ for some $f \in \Sigma_m$, $m \geq 0$, and $t_1, \ldots, t_m \in sbt(R \cup S)$. Hence the stub equation

$$f([t_1]_{\leftrightarrow^*_{R\cup S}}, \ldots, [t_m]_{\leftrightarrow^*_{R\cup S}}) \approx [t]_{\leftrightarrow^*_{R\cup S}} \tag{40}$$

is in $STN(\leftrightarrow^*_{R\cup S})$. By (b), stub equation (40) is in $STN(\leftrightarrow^*_R) \cup STN(\leftrightarrow^*_S)$. Hence $[t]_{\leftrightarrow^*_R}$ is in $stub(\leftrightarrow^*_R) \cup stub(\leftrightarrow^*_S)$. Thus (39) holds.

By (35) and (39), we have (34). By (34) and Theorem 5.2, Condition (a) holds. □

Lemma 5.5. *For any reduced GTRS R, and $p, t \in T_\Sigma$, Conditions (i) and (ii) are equivalent.*

(i) $p \leftrightarrow^*_R t$.

(ii) There are $n \geq 0$, $u \in C_\Sigma(X_n)$, $p_1, \ldots, p_n \in trunk(\leftrightarrow^*_R)$ and $t_1, \ldots, t_n \in trunk(\leftrightarrow^*_R)$ such that $p = u[p_1, \ldots, p_n]$ and $t = u[t_1, \ldots, t_n]$ and for each $i = 1, \ldots, n$, $p_i \leftrightarrow^*_R t_i$.

Proof. Assume that (i) holds. Then there are $n \geq 0$, $u \in C_\Sigma(X_n)$, $p_1, \ldots, p_n \in T_\Sigma$ and $t_1, \ldots, t_n \in T_\Sigma$ and $w_1, \ldots, w_n \in lhs(R)$ such that $p = u[p_1, \ldots, p_n]$ and $t = u[t_1, \ldots, t_n]$ and for each $i = 1, \ldots, n$, $p_i \leftrightarrow^*_R w_i \leftrightarrow^*_R t_i$. By Proposition 3.8, $p_1, \ldots, p_n \in trunk(\leftrightarrow^*_R)$ and $t_1, \ldots, t_n \in trunk(\leftrightarrow^*_R)$.

Apparently, (ii) implies (i). □

Theorem 5.6. *Let R and S be GTRSs such that $\leftrightarrow^*_R$ and $\leftrightarrow^*_S$ intersect with respect to their stubs. Then for any $p, t \in T_\Sigma$, $p \leftrightarrow^*_{R\cup S} t$ if and only if there are $n \geq 0$, $u \in C_\Sigma(X_n)$, $p_1, \ldots, p_n \in T_\Sigma$, and $t_1, \ldots, t_n \in T_\Sigma$ such that*

(i) $p = u[p_1, \ldots, p_n]$, $t = u[t_1, \ldots, t_n]$, and

(ii) for each $i = 1, \ldots, n$, $p_i \leftrightarrow^*_R t_i$ or $p_i \leftrightarrow^*_S t_i$.

Proof. ($\Rightarrow$) By Proposition 2.8, we may assume that GTRSs R and S are reduced. Let $p, t \in T_\Sigma$ such that $p \leftrightarrow^*_{R\cup S} t$. Then there are $n \geq 0$, $u \in C_\Sigma(X_n)$, $p_1, \ldots, p_n \in T_\Sigma$, and $t_1, \ldots, t_n \in T_\Sigma$, and $w_1, \ldots, w_n \in lhs(R \cup S)$ such that

- $p = u[p_1, \ldots, p_n]$, $t = u[t_1, \ldots, t_n]$, and
- for each $i = 1, \ldots, m$, $p_i \leftrightarrow^*_{R\cup S} w_i \leftrightarrow^*_{R\cup S} t_i$.

By Proposition 3.8, for each $i = 1, \ldots, n$, if $w_i \in lhs(R)$, then $w_i \in trunk(\leftrightarrow^*_R)$, otherwise $w_i \in trunk(\leftrightarrow^*_S)$. By (i), Lemma 5.1, for each $i = 1, \ldots, n$, if $w_i \in lhs(R)$ then $p_i \leftrightarrow^*_R t_i$, otherwise $p_i \leftrightarrow^*_S t_i$.

($\Leftarrow$) Let $p, t \in T_\Sigma$. Assume that there are $n \geq 0$, $u \in C_\Sigma(X_n)$, $p_1, \ldots, p_n \in T_\Sigma$, and $t_1, \ldots, t_n \in T_\Sigma$ such that (i) and (ii) hold. Then $p \leftrightarrow^*_{R\cup S} t$. □

Theorem 5.7. *For any GTRSs R, S, and V, if any two of $\leftrightarrow_R^*$, $\leftrightarrow_S^*$, and $\leftrightarrow_V^*$ intersect with respect to their stubs, then $\leftrightarrow_{R\cup S}^*$ and $\leftrightarrow_V^*$ intersect with respect to their stubs.*

Proof. Let $Z_1 \in stub(\leftrightarrow_{R\cup S}^*)$ and $Z_2 \in stub(\leftrightarrow_V^*)$ be arbitrary. By Theorem 5.2, $Z_1 \in stub(\leftrightarrow_R^*) \cup stub(\leftrightarrow_S^*)$.

First assume that $Z_1 \in stub(\leftrightarrow_R^*)$. Since $\leftrightarrow_R^*$ and $\leftrightarrow_V^*$ intersect with respect to their stubs, $Z_1 \cap Z_2 = \emptyset$ or $Z_1 = Z_2$.

Second, assume that $Z_1 \in stub(\leftrightarrow_S^*)$. This case is symmetrical to the first case. We get that $Z_1 \cap Z_2 = \emptyset$ or $Z_1 = Z_2$.

Thus $\leftrightarrow_{R\cup S}^*$ and $\leftrightarrow_V^*$ intersect with respect to their stubs. □

Theorem 5.8. *For any GTRSs R, S, and V, if any two of $\leftrightarrow_R^*$, $\leftrightarrow_S^*$, and $\leftrightarrow_V^*$ intersect with respect to their stubs, then*

$stub(\leftrightarrow_{R\cup S\cup V}^*) = stub(\leftrightarrow_R^*) \cup stub(\leftrightarrow_S^*) \cup stub(\leftrightarrow_V^*)$, *and*

$STN(\leftrightarrow_{R\cup S\cup V}^*) = STN(\leftrightarrow_R^*) \cup STN(\leftrightarrow_S^*) \cup STN(\leftrightarrow_V^*)$.

Proof. By Theorems 5.7 and 5.2,

$stub(\leftrightarrow_{R\cup S\cup V}^*) = stub(\leftrightarrow_{R\cup S}^*) \cup stub(\leftrightarrow_V^*) = stub(\leftrightarrow_R^*) \cup stub(\leftrightarrow_S^*) \cup stub(\leftrightarrow_V^*)$.

By Theorems 5.7 and 5.4,

$STN(\leftrightarrow_{R\cup S\cup V}^*) = STN(\leftrightarrow_{R\cup S}^*) \cup STN(\leftrightarrow_V^*) = STN(\leftrightarrow_R^*) \cup STN(\leftrightarrow_S^*) \cup STN(\leftrightarrow_V^*)$. □

Lemma 5.9. *Let R, S, and V be GTRSs such that any two of $\leftrightarrow_R^*$, $\leftrightarrow_S^*$, and $\leftrightarrow_V^*$ intersect with respect to their stubs. For any $p \in trunk(\leftrightarrow_R^*)$ and $t \in T_\Sigma$, if $p \leftrightarrow_{R\cup S\cup V}^* t$, then $p \leftrightarrow_R^* t$.*

Proof. We may assume that R, S, and V are reduced GTRS. By (i), Lemma 5.1, we have the lemma. □

Theorem 5.10. *Let R, S, and V be GTRSs such that any two of $\leftrightarrow_R^*$, $\leftrightarrow_S^*$, and $\leftrightarrow_V^*$ intersect with respect to their stubs. Then for any $p, t \in T_\Sigma$, $p \leftrightarrow_{R\cup S\cup V}^* t$ if and only if there are $k \geq 0$, $u \in C_\Sigma(X_k)$, $p_1, \ldots, p_k \in T_\Sigma$, and $t_1, \ldots, t_k \in T_\Sigma$ such that*

(i) $p = u[p_1, \ldots, p_k]$, $t = u[t_1, \ldots, t_k]$, and

(ii) for each $i = 1, \ldots, k$, $p_i \leftrightarrow_R^* t_i$, $p_i \leftrightarrow_S^* t_i$, or $p_i \leftrightarrow_V^* t_i$.

Proof. ($\Rightarrow$) By Proposition 2.8, we may assume that GTRSs R, S, and V are reduced. Let $p, t \in T_\Sigma$ such that $p \leftrightarrow_{R\cup S\cup V}^* t$. Then there are $k \geq 0$, $u \in C_\Sigma(X_k)$, $p_1, \ldots, p_k \in T_\Sigma$, and $t_1, \ldots, t_k \in T_\Sigma$, and $w_1, \ldots, w_k \in lhs(R \cup S \cup V)$ such that

- $p = u[p_1, \ldots, p_k]$, $t = u[t_1, \ldots, t_k]$, and
- for each $i = 1, \ldots, k$, $p_i \leftrightarrow_{R\cup S\cup V}^* w_i \leftrightarrow_{R\cup S\cup V}^* t_i$.

By Proposition 3.8, for each $i = 1, \ldots, k$,

if $w_i \in lhs(R)$, then $w_i \in trunk(\leftrightarrow_R^*)$,

if $w_i \in lhs(S)$, then $w_i \in trunk(\leftrightarrow^*_S)$, and
if $w_i \in lhs(V)$, then $w_i \in trunk(\leftrightarrow^*_V)$.
By Lemma 5.9, for each $i = 1, \dots, k$,
if $w_i \in lhs(R)$ then $p_i \leftrightarrow^*_R t_i$,
if $w_i \in lhs(S)$ then $p_i \leftrightarrow^*_S t_i$, and
if $w_i \in lhs(V)$ then $p_i \leftrightarrow^*_V t_i$.
($\Leftarrow$) Let $p, t \in T_\Sigma$. Assume that there are $k \geq 0$, $u \in C_\Sigma(X_k)$, $p_1, \dots, p_k \in T_\Sigma$, and $t_1, \dots, t_k \in T_\Sigma$ such that (i) and (ii) hold. Then $p \leftrightarrow^*_{R \cup S \cup V} t$. □

We now generalize Theorems 5.7-5.10.

Theorem 5.11. *Let $n \geq 2$ and $R1, R2, \dots, Rn$ be GTRSs such that any two of $\leftrightarrow^*_{R1}, \dots, \leftrightarrow^*_{Rn}$ intersect with respect to their stubs. Then*
(i) $\leftrightarrow^*_{R1\cup\cdots\cup Rn-1}$ and $\leftrightarrow^*_{Rn}$ intersect with respect to their stubs,
(ii) $stub(\leftrightarrow^*_{R1\cup\cdots\cup Rn}) = stub(\leftrightarrow^*_{R1}) \cup \cdots \cup stub(\leftrightarrow^*_{Rn})$, and
(iii) $STN(\leftrightarrow^*_{R1\cup\cdots\cup Rn}) = STN(\leftrightarrow^*_{R1}) \cup \cdots \cup STN(\leftrightarrow^*_{Rn})$.

Proof. We proceed by induction on n.
Base case: $n = 2$. By the assumptions of the theorem, $\leftrightarrow^*_{R1}$ and $\leftrightarrow^*_{R2}$ intersect with respect to their stubs. By Theorems 5.2 and 5.4, (ii) and (iii) hold.
Induction step: Let $n \geq 3$ and assume that the theorem holds for $n-1$. We now show that the theorem holds for n. First we show (i). Let $Z_1 \in stub(\leftrightarrow^*_{R1\cup\cdots\cup Rn-1})$ and $Z_2 \in stub(\leftrightarrow^*_{Rn})$ be arbitrary. By (ii) of the induction hypothesis, $Z_1 \in stub(\leftrightarrow^*_{R1}) \cup \cdots \cup stub(\leftrightarrow^*_{Rn-1})$. Then $Z_1 \in stub(\leftrightarrow^*_{Ri})$ for some $1 \leq i \leq n-1$. Since $\leftrightarrow^*_{Ri}$ and $\leftrightarrow^*_{Rn}$ intersect with respect to their stubs, $Z_1 \cap Z_2 = \emptyset$ or $Z_1 = Z_2$. Thus $\leftrightarrow^*_{R1\cup\cdots\cup Rn-1}$ and $\leftrightarrow^*_{Rn}$ intersect with respect to their stubs.
We now show (ii). By (i) of the induction hypothesis, Theorem 5.2, and (ii) of the induction hypothesis,
$stub(\leftrightarrow^*_{R1\cup\cdots\cup Rn}) = stub(\leftrightarrow^*_{R1\cup\cdots\cup Rn-1}) \cup stub(\leftrightarrow^*_{Rn}) =$
$stub(\leftrightarrow^*_{R1}) \cup \cdots \cup stub(\leftrightarrow^*_{Rn})$.
We now show (iii). By (i) of the induction hypothesis, Theorem 5.4, and (iii) of the induction hypothesis,
$STN(\leftrightarrow^*_{R1\cup\cdots\cup Rn}) = STN(\leftrightarrow^*_{R1\cup\cdots\cup Rn-1}) \cup STN(\leftrightarrow^*_{Rn}) =$
$STN(\leftrightarrow^*_{R1}) \cup \cdots \cup STN(\leftrightarrow^*_{Rn})$. □

Lemma 5.12. *Let $n \geq 2$ and $R1, R2, \dots, Rn$ be GTRSs such that any two of $\leftrightarrow^*_{R1}, \dots, \leftrightarrow^*_{Rn}$ intersect with respect to their stubs. For any $1 \leq i \leq n$ and $p \in trunk(\leftrightarrow^*_{Ri})$ and $t \in T_\Sigma$, if $p \leftrightarrow^*_{R1\cup\ldots\cup Rn} t$, then $p \leftrightarrow^*_{Ri} t$.*

Proof. We may assume that $R1, R2, \dots, Rn$ are reduced GTRSs and that $i = 1$. By (i), Theorem 5.11, $\leftrightarrow^*_{R1}$ and $\leftrightarrow^*_{R2\cup\ldots\cup Rn}$ intersect with respect to their stubs. By (i), Lemma 5.1, we have the lemma. □

Theorem 5.13. *Let $n \geq 2$ and $R1, R2, \ldots, Rn$ be GTRSs such that any two of $\leftrightarrow^*_{R1}, \ldots, \leftrightarrow^*_{Rn}$ intersect with respect to their stubs. Then for any $p, t \in T_\Sigma$, $p \leftrightarrow^*_{R1\cup\cdots\cup Rn} t$ if and only if there are $k \geq 0$, $u \in C_\Sigma(X_k)$, $p_1, \ldots, p_k \in T_\Sigma$, and $t_1, \ldots, t_k \in T_\Sigma$ such that*

(i) $p = u[p_1, \ldots, p_k]$, $t = u[t_1, \ldots, t_k]$, and

(ii) for each $j = 1, \ldots, k$, there is $1 \leq i \leq n$ such that $p_j \leftrightarrow^*_{Ri} t_j$.

Proof. ($\Rightarrow$) By Proposition 2.8, we may assume that GTRSs $R1, R2, \ldots, Rn$ are reduced. Let $p, t \in T_\Sigma$ such that $p \leftrightarrow^*_{R1\cup\cdots\cup Rn} t$. Then there are $k \geq 0$, $u \in C_\Sigma(X_k)$, $p_1, \ldots, p_k \in T_\Sigma$, and $t_1, \ldots, t_k \in T_\Sigma$, and $w_1, \ldots, w_k \in lhs(R1 \cup \ldots \cup Rn)$ such that

- $p = u[p_1, \ldots, p_k]$, $t = u[t_1, \ldots, t_k]$, and
- for each $j = 1, \ldots, k$, $p_j \leftrightarrow^*_{R1\cup\cdots\cup Rn} w_j \leftrightarrow^*_{R1\cup\cdots\cup Rn} t_j$.

By Proposition 3.8, for each $j = 1, \ldots, k$, if $w_j \in lhs(Ri)$ for some $1 \leq i \leq n$, then $w_j \in trunk(\leftrightarrow^*_{Ri})$. By Lemma 5.12, for each $j = 1, \ldots, k$, if $w_j \in lhs(Ri)$ for some $1 \leq i \leq n$, then $p_j \leftrightarrow^*_{Ri} t_j$. Consequently, (i) and (ii) hold.

($\Leftarrow$) Let $p, t \in T_\Sigma$. Assume that there are $k \geq 0$, $u \in C_\Sigma(X_k)$, $p_1, \ldots, p_k \in T_\Sigma$, and $t_1, \ldots, t_k \in T_\Sigma$ such that (i) and (ii) hold. Then $p \leftrightarrow^*_{R1\cup\cdots\cup Rn} t$. □

Theorem 5.14. *Let $n \geq 2$ and $R1, R2, \ldots, Rn$ be GTRSs such that any two of $\leftrightarrow^*_{R1}, \ldots, \leftrightarrow^*_{Rn}$ intersect with respect to their stubs. Let REP be a set of representatives for $stub(\leftrightarrow^*_{R1\cup\cdots\cup Rn})$. For each $i = 1, 2, \ldots, n$, let $REPi = REP \cap trunk(\leftrightarrow^*_{Ri})$. Then for each $i = 1, 2, \ldots, n$, $REPi$ is a set of representatives for $stub(\leftrightarrow^*_{Ri})$. Furthermore, for each $i = 1, 2, \ldots, n$, we can construct $REPi$.*

Proof. Let $1 \leq i \leq n$ be arbitrary. By Proposition 3.7, we can construct $REPi$.

By the definition of $REPi$, $REPi \subseteq trunk(\leftrightarrow^*_{Ri}) = \bigcup stub(\leftrightarrow^*_{Ri})$.

Let $t \in REPi$ and let s be a subtree of t. Recall that $t \in trunk(\leftrightarrow^*_{Ri})$. By Proposition 2.1, $s \in trunk(\leftrightarrow^*_{Ri})$. By Definition 3.1, $s \in REP$ as well. Thus $s \in REPi$. We get that $REPi$ is closed under subtrees.

By (ii), Theorem 5.11, $stub(\leftrightarrow^*_{R1\cup\cdots\cup Rn}) = stub(\leftrightarrow^*_{R1}) \cup \cdots \cup stub(\leftrightarrow^*_{Rn})$. Hence, each class $Z \in stub(\leftrightarrow^*_{Ri})$ contains exactly one tree $t \in REP$. By the definition of $REPi$, $t \in REPi$ as well. Consequently, each class $Z \in trunk(\leftrightarrow^*_{Ri})$ contains at least one element of $REPi$. By the definition of $REPi$, $REPi \subseteq REP$. Thus each class $Z \in stub(\leftrightarrow^*_{Ri})$ contains exactly one element of $REPi$. Therefore, for each $i = 1, 2, \ldots, n$, $REPi$ is a set of representatives for $stub(\leftrightarrow^*_{Ri})$. □

Theorem 5.15. *Let $n \geq 2$ and $R1, R2, \ldots, Rn$ be GTRSs such that any two of $\leftrightarrow^*_{R1}, \ldots, \leftrightarrow^*_{Rn}$ intersect with respect to their stubs. For each $i = 1, 2, \ldots, n$, let $REPi$ be a set of representatives for $stub(\leftrightarrow^*_{Ri})$ such that for all $Z \in stub(\leftrightarrow^*_{R1}) \cup \cdots \cup stub(\leftrightarrow^*_{Rn})$ and $s, t \in REP1 \cup \cdots \cup REPn$, if $s, t \in Z$, then $s = t$. Then $REP1 \cup \cdots \cup REPn$ is a set of representatives for $stub(\leftrightarrow^*_{R1\cup\cdots\cup Rn})$.*

Proof. By the conditions of the theorem, $REP1 \cup \cdots \cup REPn \subseteq stub(\leftrightarrow^*_{R1}) \cup \cdots \cup stub(\leftrightarrow^*_{Rn})$. By (ii), Theorem 5.11,

$$stub(\underset{R1\cup\cdots\cup Rn}{\overset{*}{\leftrightarrow}}) = stub(\underset{R1}{\overset{*}{\leftrightarrow}}) \cup \cdots \cup stub(\underset{Rn}{\overset{*}{\leftrightarrow}})\,. \tag{41}$$

Consequently, $REP1 \cup \cdots \cup REPn \subseteq stub(\leftrightarrow^*_{R1\cup\cdots\cup Rn})$.

Let $t \in REP1 \cup \cdots \cup REPn$ and let s be a subtree of t. Then $t \in REPi$ for some $1 \leq i \leq n$. By Definition 3.1, $s_i \in REPi$ as well. Thus $s \in REP1 \cup \cdots \cup REPn$. We get that $REP1 \cup \cdots \cup REPn$ is closed under subtrees.

Let $Z \in stub(\leftrightarrow^*_{R1\cup\cdots\cup Rn})$. By (41), $Z \in stub(\leftrightarrow^*_{R1}) \cup \cdots \cup stub(\leftrightarrow^*_{Rn})$. Consequently, Z contains an element of $REP1 \cup \cdots \cup REPn$. Assume that $s, t \in Z$ and $s, t \in REP1 \cup \cdots \cup REPn$. By the assumptions of the theorem, $s = t$. Thus Z contains exactly one element of $REP1 \cup \cdots \cup REPn$. □

Theorem 5.16. *Let $n \geq 2$ and $R1, R2, \ldots, Rn$ be GTRSs such that any two of $\leftrightarrow^*_{R1}, \ldots, \leftrightarrow^*_{Rn}$ intersect with respect to their stubs. Let V be a reduced GTRS such that $\leftrightarrow^*_V = \leftrightarrow^*_{R1\cup\cdots\cup Rn}$. Then we can construct the reduced GTRSs $V1, V2, \ldots, Vn$ such that $V = V1 \cup \cdots \cup Vn$ and for each $i = 1, 2, \ldots, n$, $\leftrightarrow^*_{Ri} = \leftrightarrow^*_{Vi}$.*

Proof. By Corollary 3.15,

(i) $sbt(V) - lhs(V)$ is a set of representatives for $stub(\leftrightarrow^*_V)$, and

(ii) the GTRS determined by $\leftrightarrow^*_V$, $stub(\leftrightarrow^*_V)$, and $sbt(V) - lhs(V)$ is equal to V.

For each $i = 1, 2, \ldots, n$, let $REPi = (sbt(V) - lhs(V)) \cap trunk(\leftrightarrow^*_{Ri})$. By Proposition 3.7, we can construct $REPi$ for $i = 1, 2, \ldots, n$. By Theorem 5.14, for each $i = 1, 2, \ldots, n$, $REPi$ is a set of representatives for $stub(\leftrightarrow^*_{Ri})$. For each $i = 1, 2, \ldots, n$, let Vi be the GTRS determined by $\leftrightarrow^*_{Ri}$, $stub(\leftrightarrow^*_{Ri})$, and $REPi$. By Proposition 3.4 for each $i = 1, 2, \ldots, n$, GTRS Vi is reduced and equivalent to Ri. By Proposition 3.10, we can construct the reduced GTRS Vi for $i = 1, 2, \ldots, n$.

We now show that $V = V1 \cup \cdots \cup Vn$. First we show that $V \subseteq V1 \cup \cdots \cup Vn$. Let $p \to q$ be an arbitrary rule in V. By Definition 3.2,

- $p = f(p_1, \ldots, p_m)$ for some $m \geq 0$, $f \in \Sigma_m$, and $p_1, \ldots, p_m \in sbt(V) - lhs(V)$,
- $q \in sbt(V) - lhs(V)$,
- $p \neq q$ and $p \leftrightarrow^*_V q$.

Hence

$$f^{\leftrightarrow^*_V}([p_1]_{\leftrightarrow^*_V}, \ldots, [p_m]_{\leftrightarrow^*_V}) = [p]_{\leftrightarrow^*_V} \tag{42}$$

is a $comp(\leftrightarrow^*_V)$ equality. Consequently, (42) is a $stub(\leftrightarrow^*_V)$ equality. Then

$$f([p_1]_{\leftrightarrow^*_V}, \ldots, [p_m]_{\leftrightarrow^*_V}) \approx [p]_{\leftrightarrow^*_V} \tag{43}$$

is a $stub(\leftrightarrow^*_V)$ equation. By Theorem 5.11, (43) is a $stub(\leftrightarrow^*_{Vk})$ equality for some $k \in \{1, \ldots, n\}$. Hence $p \leftrightarrow^*_{Vk} q$ and $p_1, \ldots, p_m \in stub(\leftrightarrow^*_{Vk})$ and $q \in stub(\leftrightarrow^*_{Vk})$. By the definition of $REPk$, we have $p_1, \ldots, p_m \in REPk$ and $q \in REPk$. By Definition 3.2, $p \to q$ is in Vk.

Second we show that $V1 \cup \cdots \cup Vn \subseteq V$. Let $k \in \{1, \ldots, n\}$ be arbitrary. Let $p \to q$ be an arbitrary rule in Vk. By Definition 3.2,

- $p = f(p_1, \ldots, p_m)$ for some $m \geq 0$, $f \in \Sigma_m$, and $p_1, \ldots, p_m \in REPk$,
- $q \in REPk$,
- $p \neq q$ and $p \leftrightarrow^*_{Vk} q$.

Hence

$$f^{\leftrightarrow^*_{Vk}}([p_1]_{\leftrightarrow^*_{Vk}}, \ldots, [p_m]_{\leftrightarrow^*_{Vk}}) = [p]_{\leftrightarrow^*_{Vk}} \tag{44}$$

is a $comp(\leftrightarrow^*_{Vk})$ equality. Consequently, (44) is a $stub(\leftrightarrow^*_{Vk})$ equality. Therefore

$$f([p_1]_{\leftrightarrow^*_{Vk}}, \ldots, [p_m]_{\leftrightarrow^*_{Vk}}) \approx [p]_{\leftrightarrow^*_{Vk}} \tag{45}$$

is a $stub(\leftrightarrow^*_{Vk})$ equation. By Theorem 5.11, (45) is a $stub(\leftrightarrow^*_V)$ equation. Hence $p \leftrightarrow^*_V q$. Furthermore, by the definition of $REPk$, we have $p_1, \ldots, p_m \in REP$ and $q \in REP$. By Definition 3.2, $p \to q$ is in V. □

In the light of Theorem 5.15 we state our result.

Theorem 5.17. *Let $n \geq 2$ and $R1, R2, \ldots, Rn$ be GTRSs such that any two of $\leftrightarrow^*_{R1}, \ldots, \leftrightarrow^*_{Rn}$ intersect with respect to their stubs. For each $i = 1, 2, \ldots, n$, let $REPi$ be a set of representatives for $stub(\leftrightarrow^*_{Ri})$ such that for all $Z \in stub(\leftrightarrow^*_{R1}) \cup \cdots \cup stub(\leftrightarrow^*_{Rn})$ and $s, t \in REP1 \cup \cdots \cup REPn$, if $s, t \in Z$, then $s = t$. For each $i = 1, 2, \ldots, n$, let Vi be the reduced GTRS determined by $\leftrightarrow^*_{Ri}$, $stub(\leftrightarrow^*_{Ri})$, and $REPi$. Let V be the reduced GTRS determined by $\leftrightarrow^*_{R1\cup\cdots\cup Rn}$, $stub(\leftrightarrow^*_{R1\cup\cdots\cup Rn})$, and $REP1 \cup \cdots \cup REPn$. Then $V = V1 \cup \cdots \cup Vn$. Moreover, we can construct V and Vi for $i = 1, 2, \ldots, n$.*

Proof. By Proposition 3.10, we can construct V and Vi for $i = 1, 2, \ldots, n$. First we show that $V \subseteq V1 \cup \cdots \cup Vn$. Let $p \to q$ be an arbitrary rule in V. By Definition 3.2,

- $p = f(p_1, \ldots, p_m)$ for some $m \geq 0$, $f \in \Sigma_m$, and $p_1, \ldots, p_m \in REP1 \cup \cdots \cup REPn$,
- $q \in REP1 \cup \cdots \cup REPn$,
- $p \neq q$ and $p \leftrightarrow^*_{R1\cup\cdots\cup Rn} q$.

Hence

$$f^{\leftrightarrow^*_{R1\cup\cdots\cup Rn}}([p_1]_{\leftrightarrow^*_{R1\cup\cdots\cup Rn}}, \ldots, [p_m]_{\leftrightarrow^*_{R1\cup\cdots\cup Rn}}) = [q]_{\leftrightarrow^*_{R1\cup\cdots\cup Rn}} \tag{46}$$

is a $comp(\leftrightarrow^*_{R1\cup\cdots\cup Rn})$ equality. Then (46) is a $stub(\leftrightarrow^*_{R1\cup\cdots\cup Rn})$ equality. Thus

$$f([p_1]_{\leftrightarrow^*_{R1\cup\cdots\cup Rn}}, \ldots, [p_m]_{\leftrightarrow^*_{R1\cup\cdots\cup Rn}}) \approx [q]_{\leftrightarrow^*_{R1\cup\cdots\cup Rn}} \tag{47}$$

is a $stub(\leftrightarrow^*_{R1\cup\cdots\cup Rn})$ equation. By (iii), Theorem 5.11, (47) is a $stub(\leftrightarrow^*_{Rk})$ equation for some $k \in \{1, \ldots, n\}$. Hence $p \leftrightarrow^*_{Rk} q$ and $p_1, \ldots, p_m \in stub(\leftrightarrow^*_{Rk})$ and $q \in stub(\leftrightarrow^*_{Rk})$. Hence $p_1, \ldots, p_m \in REPk$ and $q \in REPk$. By Definition 3.2, $p \to q$ is in Vk.

Second we show that $V1 \cup \cdots \cup Vn \subseteq V$. Let $k \in \{1, \ldots, n\}$ be arbitrary. Let $p \to q$ be an arbitrary rule in Vk. By Definition 3.2,

- $p = f(p_1, \ldots, p_m)$ for some $m \geq 0$, $f \in \Sigma_m$, and $p_1, \ldots, p_m \in REPk$,
- $q \in REPk$,
- $p \neq q$ and $p \leftrightarrow^*_{Vk} q$.

Hence

$$f^{\leftrightarrow^*_{Vk}}([p_1]_{\leftrightarrow^*_{Vk}}, \ldots, [p_m]_{\leftrightarrow^*_{Vk}}) = [p]_{\leftrightarrow^*_{Vk}} \tag{48}$$

is a $comp(\leftrightarrow^*_{Vk})$ equality. Therefore (48) is a $stub(\leftrightarrow^*_{Vk})$ equality. Consequently

$$f([p_1]_{\leftrightarrow^*_{Vk}}, \ldots, [p_m]_{\leftrightarrow^*_{Vk}}) = [p]_{\leftrightarrow^*_{Vk}} \tag{49}$$

is a $trunk(\leftrightarrow^*_{Vk})$ equation. By (iii), Theorem 5.11, (49) is a $stub(\leftrightarrow^*_{V})$ equation. Hence $p \leftrightarrow^*_{V} q$. Apparently, $p_1, \ldots, p_m \in REP1 \cup \cdots \cup REPn$ and $q \in REP1 \cup \cdots \cup REPn$. By Definition 3.2, $p \to q$ is in V. □

6 Elementary correspondences

In this section we show eight elementary connections between a reduced GTRS R and the algebraic constructs associated with the congruence $\leftrightarrow^*_R$. We show that for any equivalent reduced GTRSs R and S, the same number of terms appear as subterms in R as in S. We give an upper bound on the number of reduced GTRSs equivalent to a given reduced GTRS R. We show that for any convergent GTRS R, one can construct an equivalent reduced GTRS V such that $\to_V \subseteq \to^*_R$.

In the light of Theorem 3.14, we can state the following result.

Theorem 6.1. *Let R be a reduced GTRS. Then Conditions (i)-(viii) hold.*

(i) $IRR(R) \cap trunk(\leftrightarrow^*_R) = sbt(R) - lhs(R)$.

(ii) $card(lhs(R)) = card(R)$.

(iii) $card(stub(\leftrightarrow_R^*)) = card(sbt(R)) - card(R)$.

(iv) $sbt(R) - lhs(R)$ *is a set of representatives for* $stub(\leftrightarrow_R^*)$. *Each tree in* $rhs(R)$ *is a representative for a compound class. Each tree in* $sbt(R) - (lhs(R) \cup rhs(R))$ *is a representative for a class in* $simp(\leftrightarrow_R^*) \cap stub(\leftrightarrow_R^*)$. *For each class* $Z \in simp(\leftrightarrow_R^*) \cap stub(\leftrightarrow_R^*)$, $Z \cap sbt(R) = \{\, t \,\}$, *where* $t \in sbt(R) - (lhs(R) \cup rhs(R))$ *is the representative for* Z.

(v) $card(sbt(R)) = card(simp(\leftrightarrow_R^*) \cap stub(\leftrightarrow_R^*)) + card(COY) =$
$card(stub(\leftrightarrow_R^*)) + card(R)$.

(vi) $card(comp(\leftrightarrow_R^*)) = card(rhs(R))$.

(vii) $card(simp(\leftrightarrow_R^*) \cap stub(\leftrightarrow_R^*)) = card(sbt(R)) - card(lhs(R)) - card(rhs(R))$.

(viii) $card(R) = card(COY) - card(rhs(R))$.

Proof. By Proposition 3.8,

$$stub(\overset{*}{\underset{R}{\leftrightarrow}}) = [sbt(R)]_{\leftrightarrow_R^*} . \tag{50}$$

By Theorem 3.14,

(a) $sbt(R) - lhs(R)$ is a set of representatives for $[sbt(R)]_{\leftrightarrow_R^*}$, and

(b) the GTRS determined by $\leftrightarrow_R^*$, $[sbt(R)]_{\leftrightarrow_R^*}$, and $sbt(R) - lhs(R)$ is equal to R.

We now show (i). As R is reduced,

$$sbt(R) - lhs(R) \subseteq IRR(R) . \tag{51}$$

By Proposition 3.6 and (a),

$$sbt(R) - lhs(R) \subseteq trunk(\overset{*}{\underset{R}{\leftrightarrow}}) .$$

Thus

$$sbt(R) - lhs(R) \subseteq IRR(R) \cap trunk(\overset{*}{\underset{R}{\leftrightarrow}}) .$$

Conversely, let $t \in sbt(R)$ be arbitrary. By (a), the congruence class $[t]_{\leftrightarrow_R^*}$ contains a tree s in $sbt(R) - lhs(R)$. By (51), $s \in IRR(R)$. Since R is reduced, by Proposition 2.7, R is convergent. Thus the congruence class $[t]_{\leftrightarrow_R^*}$ contains exactly one tree in $IRR(R)$. Hence $IRR(R) \cap [t]_{\leftrightarrow_R^*} = \{\, s \,\}$. Thus for each $t \in sbt(R)$, $IRR(R) \cap [t]_{\leftrightarrow_R^*} \in sbt(R) - lhs(R)$. By Proposition 3.6,

$$IRR(R) \cap trunk(\overset{*}{\underset{R}{\leftrightarrow}}) \subseteq sbt(R) - lhs(R) .$$

We now show Condition (ii). By Definition 2.5, for each tree $t \in lhs(R)$, there is exactly one rule in R with left-hand side t. Hence $card(lhs(R)) = card(R)$.

We now show Condition (iii). By Definition 3.1, (50), and (a),

$$card(stub(\overset{*}{\underset{R}{\leftrightarrow}})) = card(sbt(R) - lhs(R))\,.$$

Hence Condition (ii) implies Condition (iii).

We now show Condition (iv). By (50) and (a), $sbt(R) - lhs(R)$ is a set of representatives for $stub(\leftrightarrow_R^*)$. By (50), (a) and (b), and Definitions 3.1 and 3.2,

• each tree in $rhs(R)$ is a representative for a compound class,

• each tree in $sbt(R) - (lhs(R) \cup rhs(R))$ is a representative for a class in $simp(\leftrightarrow_R^*) \cap stub(\leftrightarrow_R^*)$, and

• for each class $Z \in simp(\leftrightarrow_R^*) \cap stub(\leftrightarrow_R^*)$, $Z \cap sbt(R) = \{t\}$, where $t \in sbt(R) - (lhs(R) \cup rhs(R))$ is the representative for Z.

We now show Condition (v). By Condition (iii),

$$card(sbt(R)) = card(stub(\overset{*}{\underset{R}{\leftrightarrow}})) + card(R)\,.$$

By (iv),

$$card(sbt(R) - (lhs(R) \cup rhs(R))) = card(simp(\overset{*}{\underset{R}{\leftrightarrow}}) \cap stub(\overset{*}{\underset{R}{\leftrightarrow}}))\,.$$

Thus

$$card(sbt(R)) = card(simp(\overset{*}{\underset{R}{\leftrightarrow}}) \cap stub(\overset{*}{\underset{R}{\leftrightarrow}})) + card(lhs(R)) + card(rhs(R))\,. \quad (52)$$

Hence by (d), Lemma 4.2,

$$card(sbt(R)) = card(simp(\overset{*}{\underset{R}{\leftrightarrow}}) \cap stub(\overset{*}{\underset{R}{\leftrightarrow}})) + card(COY)\,.$$

By Condition (iii), $card(sbt(R)) = card(stub(\leftrightarrow_R^*)) + card(R)$.

Condition (iv) implies Condition (vi). Condition (vii) follows from (52). Condition (viii) is a simple consequence of (d), Lemma 4.2. □

Theorem 6.2. *For any equivalent reduced GTRSs R and S, $card(sbt(R)) = card(sbt(S))$.*

Proof. By Theorem 6.1 (v),

$$card(sbt(R)) = card(simp(\overset{*}{\underset{R}{\leftrightarrow}}) \cap stub(\overset{*}{\underset{R}{\leftrightarrow}})) + card(COY(\overset{*}{\underset{R}{\leftrightarrow}}))$$

and

$$card(sbt(S)) = card(simp(\overset{*}{\underset{S}{\leftrightarrow}}) \cap stub(\overset{*}{\underset{S}{\leftrightarrow}})) + card(COY(\overset{*}{\underset{S}{\leftrightarrow}}))\,.$$

As $\leftrightarrow_R^* = \leftrightarrow_S^*$, $card(sbt(R)) = card(sbt(S))$. □

Let R be a reduced GTRS. Let REP be a set of representatives for $stub(\leftrightarrow_R^*)$. Let $t = f(t_1, \ldots, t_m)$ be an element of REP, where $[t]_{\leftrightarrow_R^*} \in comp(\leftrightarrow_R^*)$, $f \in \Sigma_m$, $m \geq 0$, $t_1, \ldots, t_m \in REP$. We assign the compound equality

$$f^{\mathbf{TA}/\leftrightarrow_R^*}([t_1]_{\leftrightarrow_R^*}), \ldots, [t_m]_{\leftrightarrow_R^*}) = [t]_{\leftrightarrow_R^*}$$

to t. $COYREP$ is the set of all compound equalities which are assigned to the elements of $REP \cap (\bigcup comp(\leftrightarrow_R^*))$.

Lemma 6.3. *Let R be a reduced GTRS. For any sets $REP1$ and $REP2$ of representatives for $stub(\leftrightarrow_R^*)$, $REP1 = REP2$ if and only if $COYREP1 = COYREP2$.*

Proof. Let $REP1$ and $REP2$ be sets of representatives for $stub(\leftrightarrow_R^*)$. Assume that $REP1 \neq REP2$. Let $t \in REP1$ be of minimal height such that $[t]_{\leftrightarrow_R^*}$ is represented by a tree $s \in REP2$ different from t. Let $t = f(t_1, \ldots, t_m)$, where $f \in \Sigma_m$, $m \geq 0$, $t_1, \ldots, t_m \in REP1$. Let $s = g(s_1, \ldots, s_n)$, where $g \in \Sigma_n$, $n \geq 0$, $s_1, \ldots, s_n \in REP2$. If $f = g$ and $m = n$, and $[t_i]_{\leftrightarrow_R^*} = [s_i]_{\leftrightarrow_R^*}$ for $1 \leq i \leq n$, then by the definition of t, $t_i = s_i$ for $1 \leq i \leq n$. Hence $t = s$, a contradiction. Hence $f \neq g$ or $f = g$, $m = n$, and $[t_i]_{\leftrightarrow_R^*} \neq [s_i]_{\leftrightarrow_R^*}$ for some $1 \leq i \leq n$. Thus $[t]_{\leftrightarrow_R^*}$ is a compound class and the compound equality $f^{\mathbf{TA}/\leftrightarrow_R^*}([t_1]_{\leftrightarrow_R^*}), \ldots, [t_m]_{\leftrightarrow_R^*}) = [t]_{\leftrightarrow_R^*}$ assigned to the tree t is different from the compound equality $g^{\mathbf{TA}/\leftrightarrow_R^*}([s_1]_{\leftrightarrow_R^*}, \ldots, [s_n]_{\leftrightarrow_R^*}) = [s]_{\leftrightarrow_R^*}$ assigned to the tree s. Hence $COYREP1 \neq COYREP2$.

Conversely, assume that $COYREP1 \neq COYREP2$. Then there are representatives $t = f(t_1, \ldots, t_m) \in REP1$ and $s = g(s_1, \ldots, s_n) \in REP2$ such that

- $f \in \Sigma_m$, $m \geq 0$, $t_1, \ldots, t_m \in REP1$,
- the compound equality
$$f^{\mathbf{TA}/\leftrightarrow_R^*}([t_1]_{\leftrightarrow_R^*}), \ldots, [t_m]_{\leftrightarrow_R^*}) = [t]_{\leftrightarrow_R^*} \tag{53}$$
is assigned to the tree t,
- $g \in \Sigma_n$, $n \geq 0$, $s_1, \ldots, s_n \in REP2$, and the compound equality
$$g^{\mathbf{TA}/\leftrightarrow_R^*}([s_1]_{\leftrightarrow_R^*}), \ldots, [s_m]_{\leftrightarrow_R^*}) = [s]_{\leftrightarrow_R^*} \tag{54}$$
is assigned to the tree s,
- $[t]_{\leftrightarrow_R^*} = [s]_{\leftrightarrow_R^*}$, and
- compound equality (53) is different from compound equality (54).

Hence $f \neq g$ or $f = g$, $m = n$, and $[t_i]_{\leftrightarrow_R^*} \neq [s_i]_{\leftrightarrow_R^*}$ for some $1 \leq i \leq m$. Hence $s \neq t$. As both s and t are the representatives of the class $[s]_{\leftrightarrow_R^*}$, $REP1 \neq REP2$. □

Theorem 6.4. *Let R be a reduced GTRS with $rhs(R) = \{ t_1, \ldots, t_n \}$, $n \geq 0$. Then there are at most $deg([t_1]_{\leftrightarrow_R^*}) \cdot deg([t_2]_{\leftrightarrow_R^*}) \cdot \ldots \cdot deg([t_n]_{\leftrightarrow_R^*})$ reduced GTRSs equivalent to R.*

Proof. When we choose a set REP of representatives for $stub(\leftrightarrow_R^*)$ and assign a set $COYREP$ of compound equalities to REP, we choose a representative t for each compound $\leftrightarrow_R^*$-class Z, and assign a compound equality to it. For each compound $\leftrightarrow_R^*$-class Z, we can assign at most $deg(Z)$ compound equalities to the representative t of Z, see Definition 3.16. Consequently, the number of sets $COYREP$ of compound equalities assigned to the sets REP of representatives for $stub(\leftrightarrow_R^*)$ is less than or equal to $deg([t_1]_{\leftrightarrow_R^*}) \cdot deg([t_2]_{\leftrightarrow_R^*}) \cdot \ldots \cdot deg([t_n]_{\leftrightarrow_R^*})$. Hence by Lemma 6.3, the number of the sets of representatives for $stub(\leftrightarrow_R^*)$ is less than or equal to $deg([t_1]_{\leftrightarrow_R^*}) \cdot deg([t_2]_{\leftrightarrow_R^*}) \cdot \ldots \cdot deg([t_n]_{\leftrightarrow_R^*})$. By Proposition 3.11 and Proposition 3.8, for each reduced GTRS R' equivalent to R, there exists a set REP of representatives for $stub(\leftrightarrow_R^*)$ such that the GTRS determined by $\leftrightarrow_R^*$, $stub(\leftrightarrow_R^*)$, and REP is equal to R'. Hence there are at most $deg([t_1]_{\leftrightarrow_R^*}) \cdot deg([t_2]_{\leftrightarrow_R^*}) \cdot \ldots \cdot deg([t_n]_{\leftrightarrow_R^*})$ reduced GTRSs R' equivalent to R.

For each integer $l \geq 1$, we have $l \leq 2^{l-1}$. Hence $deg([t_1]_{\leftrightarrow_R^*}) \cdot deg([t_2]_{\leftrightarrow_R^*}) \cdot \ldots \cdot deg([t_n]_{\leftrightarrow_R^*}) \leq 2^{SUM}$, where $SUM = deg([t_1]_{\leftrightarrow_R^*}) + deg([t_2]_{\leftrightarrow_R^*}) + \ldots + deg([t_n]_{\leftrightarrow_R^*}) - n$. Obviously,

$$card(COY) = deg([t_1]_{\leftrightarrow_R^*}) + deg([t_2]_{\leftrightarrow_R^*}) + \ldots + deg([t_n]_{\leftrightarrow_R^*}) .$$

Hence $SUM = card(COY) - n$. By the assumption of the theorem, $card(rhs(R)) = n$. By (b), Lemma 4.2, $card(COY) = card(R) + n$. Consequently, $SUM = card(R)$. Thus

$$deg([t_1]_{\leftrightarrow_R^*}) \cdot deg([t_2]_{\leftrightarrow_R^*}) \cdot \ldots \cdot deg([t_n]_{\leftrightarrow_R^*}) \leq 2^{card(R)} .$$

□

One can also show Theorem 6.4 by modifying the proof of Theorem 4.7 in [18] in the following way. One can apply Snyder's Fast Ground Completion algorithm also for a GTRS similarly as for a set of ground term equations. We apply Snyder's Fast Ground Completion algorithm for a reduced GTRS rather than a GTRS. Then the number k_i denoting the total number of vertices in the compound class $[t_i]_{\leftrightarrow_R^*}$ (called a non-trivial class in [18]) is equal to $deg([t_i]_{\leftrightarrow_R^*})$ for $1 \leq i \leq n$.

Theorem 6.5. *For any convergent GTRS R, one can effectively construct an equivalent reduced GTRS V such that $\rightarrow_V \subseteq \rightarrow_R^*$.*

Proof. Applying Snyder's [18] Fast Ground Completion algorithm for GTRS R we construct an equivalent reduced GTRS S. For each term $t \in sbt(S)$, we compute its R-normal form. Let REP be the set of all R-normal forms of the elements of $sbt(S)$. Each class $Z \in [sbt(S)]_{\leftrightarrow_S^*}$ contains exactly one tree in REP. Furthermore,

$$REP \subseteq \bigcup [sbt(S)]_{\leftrightarrow_S^*} . \tag{55}$$

By Proposition 3.8, each class $Z \in stub(\leftrightarrow_S^*)$ contains exactly one tree in REP, and $REP \subseteq \bigcup stub(\leftrightarrow_S^*)$.

We now show that REP is closed under subtrees. Let $u \in REP$ be arbitrary, and let $v \in sbt(u)$. By (55) and Proposition 3.6, $u \in trunk(\leftrightarrow_S^*)$. As $trunk(\leftrightarrow_S^*)$ is closed under subtrees, $v \in trunk(\leftrightarrow_S^*)$. By Proposition 3.6, $v \leftrightarrow_S^* w$ for some $w \in sbt(S)$. Hence $v \leftrightarrow_R^* w$. Since v is a subtree of u, v is irreducible for R. Hence v is the R-normal form of $w \in sbt(S)$. Thus $v \in REP$. We have shown that

- each class $Z \in stub(\leftrightarrow_S^*)$ contains exactly one tree in REP,
- $REP \subseteq \bigcup stub(\leftrightarrow_S^*)$, and
- REP is closed under subtrees.

By Definition 3.1, REP is a set of representatives for $stub(\leftrightarrow_S^*)$. By Definition 3.2, $\leftrightarrow_S^*$, $stub(\leftrightarrow_S^*)$, and REP determine a GTRS V. Since S is reduced, S is convergent, see Proposition 2.7. Thus for any terms $p, q \in T_\Sigma$ we can decide whether $p \leftrightarrow_S^* q$. So we can effectively construct V. Let $p \to q$ be an arbitrary rule in V. By Definition 3.2, $p \leftrightarrow_S^* q$. Hence $p \leftrightarrow_R^* q$. Since $q \in REP$ is an R-normal form, $p \to_R^* q$. Hence $\to_V \subseteq \to_R^*$. By Proposition 3.4, V is a reduced GTRS, and V is equivalent to S. Thus V is equivalent to R. □

7 Examples

We illustrate our concepts and results by examples.

Example 7.1. Let $\Sigma = \Sigma_0 \cup \Sigma_1$, $\Sigma_0 = \{a, b\}$, $\Sigma_1 = \{f\}$. Let the GTRS R consist of the rules

$a \to b$,

$f(f(b)) \to b$.

Observe that R is reduced. We have $sbt(R) = \{a, b, f(b), f(f(b))\}$.

$sbt(R) - lhs(R) = \{b, f(b)\}$.

$sbt(R) - (lhs(R) \cup rhs(R)) = \{f(b)\}$.

$trunk(\leftrightarrow_R^*) = T_\Sigma$.

$IRR(R) = IRR(R) \cap trunk(\leftrightarrow_R^*) = \{b, f(b))\}$.

$stub(\leftrightarrow^*_R) = [sbt(R)]_{\leftrightarrow^*_R} = \{ [b]_{\leftrightarrow^*_R}, [f(b)]_{\leftrightarrow^*_R} \}$.
$comp(\leftrightarrow^*_R) = \{ [b]_{\leftrightarrow^*_R} \}$.
$simp(\leftrightarrow^*_R) = simp(\leftrightarrow^*_R) \cap stub(\leftrightarrow^*_R) = \{ [f(b)]_{\leftrightarrow^*_R} \}$.
$sbt(R) - lhs(R) = \{ b, f(b) \}$ is a set of representatives for $[sbt(R)]_{\leftrightarrow^*_R}$.
$[f(b)]_{\leftrightarrow^*_R} \cap sbt(R) = \{ f(b) \}$ and $f(b) \in sbt(R) - (lhs(R) \cup rhs(R))$ is the representative for $[f(b)]_{\leftrightarrow^*_R}$.
The GTRS determined by $\leftrightarrow^*_R$, $\{ [b]_{\leftrightarrow^*_R}, [f(b)]_{\leftrightarrow^*_R} \}$, and $\{ b, f(b) \}$ is equal to R.
COY consists of the equalities

$$a^{\leftrightarrow^*_R} = [b]_{\leftrightarrow^*_R} \tag{56}$$

$$b^{\leftrightarrow^*_R} = [b]_{\leftrightarrow^*_R} \tag{57}$$

$$f^{\leftrightarrow^*_R}([f(b)]_{\leftrightarrow^*_R}) = [b]_{\leftrightarrow^*_R} . \tag{58}$$

We define the mapping $\phi : COY \to lhs(R) \cup rhs(R)$ as follows. ϕ assigns a to the equality (56). ϕ assigns b to the equality (57). ϕ assigns $f(f(b))$ to the equality (58).
STY consists of the equalities (56), (57), (58), and

$$f^{\leftrightarrow^*_R}([b]_{\leftrightarrow^*_R}) = [f(b)]_{\leftrightarrow^*_R} . \tag{59}$$

We extend the mapping ϕ to the mapping $\psi : STY \to sbt(R)$. ψ assigns $f(b)$ to the equality (59).

CON consists of the equations

$$a \approx [b]_{\leftrightarrow^*_R} \tag{60}$$

$$b \approx [b]_{\leftrightarrow^*_R} \tag{61}$$

$$f([f(b)]_{\leftrightarrow^*_R}) \approx [b]_{\leftrightarrow^*_R}. \tag{62}$$

STN consists of the equations (60), (61), (62), and

$$f([b]_{\leftrightarrow^*_R}) \approx [f(b)]_{\leftrightarrow^*_R} . \tag{63}$$

By Proposition 2.10, there are at most 2^2 reduced GTRSs equivalent to R. Observe that $deg([b]_{\leftrightarrow^*_R}) = 3$. By Theorem 6.4, there are at most $deg([b]_{\leftrightarrow^*_R}) = 3$ reduced GTRSs equivalent to R. We define the GTRS S changing the role of a and b. S consists of the rules

$b \to a$,

$f(f(a)) \to a$.

Observe that S is reduced and is equivalent to R. We now show that R and S are the only two reduced GTRSs which are equivalent to R. Let U be a reduced GTRS which is equivalent to R. By Proposition 3.8 and Theorem 3.14,

(a) $sbt(U) - lhs(U)$ is a set of representatives for $stub(\leftrightarrow_R^*)$, and

(b) the GTRS determined by $\leftrightarrow_R^*$, $stub(\leftrightarrow_R^*)$, and $sbt(U) - lhs(U)$ is equal to U.

By Definition 3.1 and Condition (a), $sbt(U) - lhs(U)$ is closed under subtrees. Therefore, $a \in sbt(U) - lhs(U)$ or $b \in sbt(U) - lhs(U)$. First assume that $a \in sbt(U) - lhs(U)$. Then $f(a)$ is the representative of $[f(a)]_{\leftrightarrow_R^*}$. Then $sbt(U) - lhs(U) = \{a, f(a)\}$. Consequently, the GTRS determined by $\leftrightarrow_R^*$, $stub(\leftrightarrow_R^*)$, and $sbt(U) - lhs(U)$ is equal to S. Hence by (b), $U = S$. Second assume that $b \in sbt(U) - lhs(U)$. Symmetrically to the first case, we obtain that $U = R$. Thus R and S are the only two reduced GTRSs which are equivalent to R.

Example 7.2. Let $\Sigma = \Sigma_0 \cup \Sigma_1$, $\Sigma_0 = \{a, b\}$, $\Sigma_1 = \{e, f, g, h\}$. Let $n \geq 1$ be arbitrary. Let the GTRS R consist of the rules

$a \to b$,

$e(f^{i-1}(b)) \to f^i(b)$ for $1 \leq i \leq n$.

Let the GTRS S consist of the rules

$a \to b$,

$g(h^{i-1}(b)) \to h^i(b)$ for $1 \leq i \leq n$.

First, we study $\leftrightarrow_R^*$. $stub(\leftrightarrow_R^*)$ consists of the following congruence classes:

$[b]_{\leftrightarrow_R^*} = \{a, b\}$,

$[f(b)]_{\leftrightarrow_R^*} = \{e(a), e(b), f(a), f(b)\}$,

$[f^2(b)]_{\leftrightarrow_R^*} = \{e^2(a), e^2(b), e(f(a)), e(f(b)), f(e(a)), f(e(b)), f(f(a)), f(f(b))\}$,

$\ldots$,

$[f^n(b)]_{\leftrightarrow_R^*} = \{e^n(a), e^n(b), e^{n-1}(f(a)), e^{n-1}(f(b)), \ldots, f^{n-1}(e(a)), f^{n-1}(e(b)),$
$f^n(a), f^n(b)\}$.

Observe that

$$comp(\overset{*}{\underset{R}{\leftrightarrow}}) = stub(\overset{*}{\underset{R}{\leftrightarrow}}).$$

$STN(\leftrightarrow_R^*)$ consists of following equations:

$a \approx [b]_{\leftrightarrow_R^*}$,

$b \approx [b]_{\leftrightarrow_R^*}$,

$e([b]_{\leftrightarrow_R^*}) \approx [f(b)]_{\leftrightarrow_R^*}$,

$f([b]_{\leftrightarrow_R^*}) \approx [f(b)]_{\leftrightarrow_R^*}$,

$e([f(b)]_{\leftrightarrow_R^*}) \approx [f^2(b)]_{\leftrightarrow_R^*}$,

$f([f(b)]_{\leftrightarrow_R^*}) \approx [f^2(b)]_{\leftrightarrow_R^*}$,

$\ldots$,

$e([f^{n-1}(b)]_{\leftrightarrow_R^*}) \approx [f^n(b)]_{\leftrightarrow_R^*}$,

$f([f^{n-1}(b)]_{\leftrightarrow_R^*}) \approx [f^n(b)]_{\leftrightarrow_R^*}$.

Apparently, $CON(\leftrightarrow_R^*) = STN(\leftrightarrow_R^*)$.

Second, we study $\leftrightarrow_S^*$. $stub(\leftrightarrow_S^*)$ consists of the following congruence classes:

$[b]_{\leftrightarrow_S^*} = \{a, b\}$,

$[h(b)]_{\leftrightarrow^*_S} = \{\, g(a), g(b), h(a), h(b) \,\}$,
$[h^2(b)]_{\leftrightarrow^*_S} = \{\, g^2(a), g^2(b), g(h(a)), g(h(b)), h(g(a)), h(g(b)), h(h(a)), h(h(b)) \,\}$,
$\ldots$,
$[h^n(b)]_{\leftrightarrow^*_S} = \{\, g^n(a), g^n(b), g^{n-1}(h(a)), g^{n-1}(h(b)), \ldots, h^{n-1}(g(a)), h^{n-1}(g(b)), h^n(a), h^n(b) \,\}$.

Observe that

$$comp(\overset{*}{\underset{S}{\leftrightarrow}}) = stub(\overset{*}{\underset{S}{\leftrightarrow}}) .$$

$STN(\leftrightarrow^*_S)$ consists of following equations:

$a \approx [b]_{\leftrightarrow^*_S}$,
$b \approx [b]_{\leftrightarrow^*_S}$,
$g([a]_{\leftrightarrow^*_S}) \approx [h(a)]_{\leftrightarrow^*_S}$,
$h([a]_{\leftrightarrow^*_S}) \approx [h(a)]_{\leftrightarrow^*_S}$,
$g([h(a)]_{\leftrightarrow^*_S}) \approx [h^2(a)]_{\leftrightarrow^*_S}$,
$h([h(a)]_{\leftrightarrow^*_S}) \approx [h^2(a)]_{\leftrightarrow^*_S}$,
$\ldots$,
$g([h^{n-1}(a)]_{\leftrightarrow^*_S}) \approx [h^n(a)]_{\leftrightarrow^*_S}$,
$h([h^{n-1}(a)]_{\leftrightarrow^*_S}) \approx [h^n(a)]_{\leftrightarrow^*_S}$.

Apparently, $CON(\leftrightarrow^*_S) = STN(\leftrightarrow^*_S)$.

Third, we study $\leftrightarrow^*_{R\cup S}$. $stub(\leftrightarrow^*_{R\cup S})$ consists of the following congruence classes:

$[b]_{\leftrightarrow^*_{R\cup S}} = \{\, a, b \,\}$,
$[f(b)]_{\leftrightarrow^*_{R\cup S}} = \{\, e(a), e(b), f(a), f(b) \,\}$,
$[f^2(b)]_{\leftrightarrow^*_{R\cup S}} = \{\, e^2(a), e^2(b), e(f(a)), e(f(b)), f(e(a)), f(e(b)), f(f(a)), f(f(b)) \,\}$,
$\ldots$,
$[f^n(b)]_{\leftrightarrow^*_{R\cup S}} = \{\, e^n(a), e^n(b), e^{n-1}(f(a)), e^{n-1}(f(b)), \ldots, f^{n-1}(e(a)), f^{n-1}(e(b)), f^n(a), f^n(b) \,\}$,
$[h(b)]_{\leftrightarrow^*_{R\cup S}} = \{\, g(a), g(b), h(a), h(b) \,\}$,
$[h^2(b)]_{\leftrightarrow^*_{R\cup S}} = \{\, g^2(a), g^2(b), g(h(a)), g(h(b)), h(g(a)), h(g(b)), h(h(a)), h(h(b)) \,\}$,
$\ldots$,
$[h^n(b)]_{\leftrightarrow^*_{R\cup S}} = \{\, g^n(a), g^n(b), g^{n-1}(h(a)), g^{n-1}(h(b)), \ldots, h^{n-1}(g(a)), h^{n-1}(g(b)), h^n(a), h^n(b) \,\}$.

Observe that

$$comp(\overset{*}{\underset{R\cup S}{\leftrightarrow}}) = stub(\overset{*}{\underset{R\cup S}{\leftrightarrow}})$$

and $STN(\leftrightarrow^*_{R\cup S}) = STN(\leftrightarrow^*_R) \cup STN(\leftrightarrow^*_S)$.

Observe that for any $Z_1 \in stub(\leftrightarrow^*_R)$ and $Z_2 \in stub(\leftrightarrow^*_S)$, if $Z_1 \cap Z_2 \neq \emptyset$, then $Z_1 = [b]_{\leftrightarrow^*_R} = Z_2 = [b]_{\leftrightarrow^*_S} = \{\, a, b \,\}$. Thus $\leftrightarrow^*_R$ and $\leftrightarrow^*_S$ intersect with respect to their stubs.

Example 7.3. Let $\Sigma = \Sigma_0 \cup \Sigma_1$, $\Sigma_0 = \{a\}$, $\Sigma_1 = \{f, g, h\}$.
Let the reduced GTRS R consist of the rules
$f(a) \to a$,
$g(g(a)) \to g(a)$.
Let the reduced GTRS S consist of the rules
$f(a) \to a$,
$h(h(a)) \to h(a)$.
Then $stub(\leftrightarrow_R^*)$ consists of the following congruence classes:
$[a]_{\leftrightarrow_R^*} = \{a, f(a), f^2(a), f^3(a), \ldots\}$,
$[g(a)]_{\leftrightarrow_R^*} = \{g(a), g^2(a), g^3(a), \ldots, g(f(a)), g^2(f(a)), g^3(f(a)), \ldots, g(f^2(a)), g^2(f^2(a)), g^3(f^2(a)), \ldots\}$.
$stub(\leftrightarrow_S^*)$ consists of the following congruence classes:
$[a]_{\leftrightarrow_S^*} = \{a, f(a), f^2(a), f^3(a), \ldots\}$,
$[h(a)]_{\leftrightarrow_S^*} = \{h(a), h^2(a), h^3(a), \ldots, h(f(a)), h^2(f(a)), h^3(f(a)), \ldots, h(f^2(a)), h^2(f^2(a)), h^3(f^2(a)), \ldots\}$.
$R \cup S$ is a reduced GTRS.
$stub(\leftrightarrow_{R\cup S}^*)$ consists of the following congruence classes:
$[a]_{\leftrightarrow_{R\cup S}^*} = \{a, f(a), f^2(a), f^3(a), \ldots\}$,
$[g(a)]_{\leftrightarrow_{R\cup S}^*} = \{g(a), g^2(a), g^3(a), \ldots, g(f(a)), g^2(f(a)), g^3(f(a)), \ldots, g(f^2(a)), g^2(f^2(a)), g^3(f^2(a)), \ldots\}$,
$[h(a)]_{\leftrightarrow_{R\cup S}^*} = \{h(a), h^2(a), h^3(a), \ldots, h(f(a)), h^2(f(a)), h^3(f(a)), \ldots, h(f^2(a)), h^2(f^2(a)), h^3(f^2(a)), \ldots\}$.
Observe that for any $Z_1 \in stub(\leftrightarrow_R^*)$ and $Z_2 \in stub(\leftrightarrow_S^*)$, if $Z_1 \cap Z_2 \neq \emptyset$, then $Z_1 = Z_2 = [a]_{\leftrightarrow_R^*} = [a]_{\leftrightarrow_S^*}$. Thus $\leftrightarrow_R^*$ and $\leftrightarrow_S^*$ intersect with respect to their stubs.
$sbt(R) - lhs(R) = \{a, g(a)\}$ is a set of representatives for $stub(\leftrightarrow_R^*)$. $\leftrightarrow_R^*$, $stub(\leftrightarrow_R^*)$, and $\{a, g(a)\}$ determine the reduced GTRS R.
$sbt(S) - lhs(S) = \{a, h(a)\}$ is a set of representatives for $stub(\leftrightarrow_S^*)$. $\leftrightarrow_S^*$, $stub(\leftrightarrow_S^*)$, and $\{a, h(a)\}$ determine the reduced GTRS S.
$(sbt(R) - lhs(R)) \cup (sbt(S) - lhs(S)) = sbt(R \cup S) - lhs(R \cup S) = \{a, g(a), h(a)\}$ is a set of representatives for $stub(\leftrightarrow_{R\cup S}^*)$.
$\leftrightarrow_{R\cup S}^*$, $stub(\leftrightarrow_{R\cup S}^*)$, and $\{a, g(a), h(a)\}$ determine the reduced GTRS $R \cup S$.
$stub(\leftrightarrow_R^*) \cup stub(\leftrightarrow_S^*) = \{[a]_{\leftrightarrow_R^*}, [g(a)]_{\leftrightarrow_R^*}, [h(a)]_{\leftrightarrow_S^*}\}$ and $\{a, g(a), h(a)\} \cap [a]_{\leftrightarrow_R^*} = \{a\}$, $\{a, g(a), h(a)\} \cap [g(a)]_{\leftrightarrow_R^*} = \{g(a)\}$, and $\{a, g(a), h(a)\} \cap [h(a)]_{\leftrightarrow_S^*}\} = \{h(a)\}$.
Hence for all $s, t \in \{a, g(a), h(a)\} = (sbt(R) - lhs(R)) \cup (sbt(S) - lhs(S))$,
if $s, t \in [a]_{\leftrightarrow_R^*}$, then $s = t = a$,
if $s, t \in [g(a)]_{\leftrightarrow_R^*}$, then $s = t = g(a)$, and
if $s, t \in [h(a)]_{\leftrightarrow_S^*}$, then $s = t = h(a)$.

We now adopt an example of Snyder [18].

Example 7.4. Let $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$, $\Sigma_0 = \{a, b, c\}$, $\Sigma_1 = \{f, h, m\}$, $\Sigma_2 = \{g\}$. Let GTRS R consist of the rules $f(f(f(a))) \to a$, $f(f(a)) \to a$, $g(c, c) \to f(a)$,

$g(c, h(a)) \rightarrow g(c, c)$, $c \rightarrow h(a)$, $b \rightarrow m(f(a))$. Snyder [18] constructed the six reduced GTRSs $R_1, R_2, R_3, R_4, R_5, R_6$ which are equivalent to R.

R_1 consists of the four rules $f(a) \rightarrow a$, $g(c, c) \rightarrow a$, $m(a) \rightarrow b$, $h(a) \rightarrow c$. The set of subterms appearing in the rules of R_1 consists of the seven terms a, b, c, $f(a)$, $g(c, c)$, $h(a)$, $m(a)$. $REP1 = \{a, b, c\}$ is a set of representatives for $[sbt(R_1)]_{\leftrightarrow_R^*}$, and the GTRS determined by $\leftrightarrow_R^*$, $[sbt(R_1)]_{\leftrightarrow_R^*}$, and $REP1$ is equal to R_1. $COYREP1$ is the set of all compound equalities which are assigned to the elements of $REP1$. $COYREP1$ consists of the following elements:

$a^{\mathbf{TA}/\leftrightarrow_R^*} = [a]_{\leftrightarrow_R^*}$,
$b^{\mathbf{TA}/\leftrightarrow_R^*} = [b]_{\leftrightarrow_R^*}$,
$c^{\mathbf{TA}/\leftrightarrow_R^*} = [c]_{\leftrightarrow_R^*}$.

$rhs(R_1) = \{a, b, c\}$. By (c), Lemma 4.2, $deg([a]_{\leftrightarrow_R^*}) = 3$, $deg([b]_{\leftrightarrow_R^*}) = 2$, and $deg([c]_{\leftrightarrow_R^*}) = 2$. By Theorem 6.4, there are at most $deg([a]_{\leftrightarrow_R^*}) \cdot deg([b]_{\leftrightarrow_R^*}) \cdot deg([c]_{\leftrightarrow_R^*}) = 3 \cdot 2 \cdot 2 = 12$ reduced GTRSs equivalent to R_1. By Snyder's example we know that there are six reduced GTRSs equivalent to R_1.

R_2 consists of the four rules $f(a) \rightarrow a$, $g(h(a), h(a)) \rightarrow a$, $m(a) \rightarrow b$, $c \rightarrow h(a)$. The set of subterms appearing in the rules of R_2 consists of the seven terms a, b, c, $f(a)$, $g(h(a), h(a))$, $h(a)$, $m(a)$. $REP2 = \{a, b, h(a)\}$ is a set of representatives for $[sbt(R_2)]_{\leftrightarrow_R^*}$, and the GTRS determined by $\leftrightarrow_R^*$, $[sbt(R_2)]_{\leftrightarrow_R^*}$, and $REP2$ is equal to R_2. $COYREP2$ is the set of all compound equalities which are assigned to the elements of $REP2$. $COYREP2$ consists of the following elements:

$a^{\mathbf{TA}/\leftrightarrow_R^*} = [a]_{\leftrightarrow_R^*}$,
$b^{\mathbf{TA}/\leftrightarrow_R^*} = [b]_{\leftrightarrow_R^*}$,
$h^{\mathbf{TA}/\leftrightarrow_R^*}([a]_{\leftrightarrow_R^*}) = [c]_{\leftrightarrow_R^*}$.

R_3 consists of the four rules $f(a) \rightarrow a$, $g(c, c) \rightarrow a$, $b \rightarrow m(a)$, $h(a) \rightarrow c$. The set of subterms appearing in the rules of R_3 consists of the seven terms a, b, c, $f(a)$, $g(c, c)$, $h(a)$, $m(a)$. $REP3 = \{a, m(a), c\}$ is a set of representatives for $[sbt(R_3)]_{\leftrightarrow_R^*}$, and the GTRS determined by $\leftrightarrow_R^*$, $[sbt(R_3)]_{\leftrightarrow_R^*}$, and $REP3$ is equal to R_3. $COYREP3$ is the set of all compound equalities which are assigned to the elements of $REP3$. $COYREP3$ consists of the following elements:

$a^{\mathbf{TA}/\leftrightarrow_R^*} = [a]_{\leftrightarrow_R^*}$,
$m^{\mathbf{TA}/\leftrightarrow_R^*}([a]_{\leftrightarrow_R^*}) = [b]_{\leftrightarrow_R^*}$,
$c^{\mathbf{TA}/\leftrightarrow_R^*} = [c]_{\leftrightarrow_R^*}$.

R_4 consists of the four rules $f(a) \rightarrow a$, $g(h(a), h(a)) \rightarrow a$, $b \rightarrow m(a)$, $c \rightarrow h(a)$. The set of subterms appearing in the rules of R_4 consists of the seven terms a, b, c, $f(a)$, $g(h(a), h(a))$, $h(a)$, $m(a)$. $REP4 = \{a, m(a), h(a)\}$ is a set of representatives for $[sbt(R_4)]_{\leftrightarrow_R^*}$, and the GTRS determined by $\leftrightarrow_R^*$, $[sbt(R_4)]_{\leftrightarrow_R^*}$, and $REP4$ is equal to R_4. $COYREP4$ is the set of all compound equalities which are assigned to the elements of $REP4$. $COYREP4$ consists of the following elements:

$a^{\mathbf{TA}/\leftrightarrow_R^*} = [a]_{\leftrightarrow_R^*}$,

$m^{\mathbf{TA}/\leftrightarrow^*_R}([a]_{\leftrightarrow^*_R}) = [b]_{\leftrightarrow^*_R}$,
$h^{\mathbf{TA}/\leftrightarrow^*_R}([a]_{\leftrightarrow^*_R}) = [c]_{\leftrightarrow^*_R}$.

R_5 consists of the four rules $f(g(c, c)) \to g(c, c)$, $a \to g(c, c)$, $m(g(c, c)) \to b$, $h(g(c, c)) \to c$. The set of subterms appearing in the rules of R_5 consists of the seven terms a, b, c, $f(g(c, c))$, $g(c, c)$, $h(g(c, c))$, $m(g(c, c))$. $REP5 = \{ b, c, g(c, c) \}$ is a set of representatives for $[sbt(R_5)]_{\leftrightarrow^*_{R_5}}$, and the GTRS determined by $\leftrightarrow^*_{R_5}$, $[sbt(R_5)]_{\leftrightarrow^*_{R_5}}$, and $REP5$ is equal to R_5. $COYREP5$ is the set of all compound equalities which are assigned to the elements of $REP5$. $COYREP5$ consists of the following elements:

$b^{\mathbf{TA}/\leftrightarrow^*_R} = [b]_{\leftrightarrow^*_R}$,
$c^{\mathbf{TA}/\leftrightarrow^*_R} = [c]_{\leftrightarrow^*_R}$,
$g^{\mathbf{TA}/\leftrightarrow^*_R}([c]_{\leftrightarrow^*_R}, [c]_{\leftrightarrow^*_R}) = [a]_{\leftrightarrow^*_R}$.

R_6 consists of the following four rules. $f(g(c, c)) \to g(c, c)$, $a \to g(c, c)$, $b \to m(g(c, c))$, $h(g(c, c)) \to c$. The set of subterms appearing in the rules of R_6 consists of the seven terms a, b, c, $f(g(c, c))$, $g(c, c)$, $h(g(c, c))$, $m(g(c, c))$. $REP6 = \{ c, g(c, c), m(g(c, c)) \}$ is a set of representatives for $[sbt(R_6)]_{\leftrightarrow^*_{R_6}}$, and the GTRS determined by $\leftrightarrow^*_{R_6}$, $[sbt(R_6)]_{\leftrightarrow^*_{R_6}}$, and $REP6$ is equal to R_6. $COYREP6$ is the set of all compound equalities which are assigned to the elements of $REP6$. $COYREP6$ consists of the following elements:

$c^{\mathbf{TA}/\leftrightarrow^*_R} = [c]_{\leftrightarrow^*_R}$,
$g^{\mathbf{TA}/\leftrightarrow^*_R}([c]_{\leftrightarrow^*_R}, [c]_{\leftrightarrow^*_R}) = [a]_{\leftrightarrow^*_R}$,
$m^{\mathbf{TA}/\leftrightarrow^*_R}([a]_{\leftrightarrow^*_R}) = [b]_{\leftrightarrow^*_R}$.

Consider the reduced GTRS R_1 once more. By Proposition 3.8,

$$stub(\overset{*}{\underset{R_1}{\leftrightarrow}}) = [sbt(R_1)]_{\leftrightarrow^*_{R_1}} .$$

Hence $stub(\leftrightarrow^*_{R_1})$ consists of the $\leftrightarrow^*_{R_1}$-classes $[a]_{\leftrightarrow^*_{R_1}}$, $[b]_{\leftrightarrow^*_{R_1}}$, $[c]_{\leftrightarrow^*_{R_1}}$. COY consists of the compound equalities

$a^{\mathbf{TA}/\leftrightarrow^*_{R_1}} = [a]_{\leftrightarrow^*_{R_1}}$,
$b^{\mathbf{TA}/\leftrightarrow^*_{R_1}} = [b]_{\leftrightarrow^*_{R_1}}$,
$c^{\mathbf{TA}/\leftrightarrow^*_{R_1}} = [c]_{\leftrightarrow^*_{R_1}}$,
$f^{\mathbf{TA}/\leftrightarrow^*_{R_1}}([a]_{\leftrightarrow^*_{R_1}}) = [a]_{\leftrightarrow^*_{R_1}}$,
$h^{\mathbf{TA}/\leftrightarrow^*_{R_1}}([a]_{\leftrightarrow^*_{R_1}}) = [c]_{\leftrightarrow^*_{R_1}}$,
$m^{\mathbf{TA}/\leftrightarrow^*_{R_1}}([a]_{\leftrightarrow^*_{R_1}}) = [b]_{\leftrightarrow^*_{R_1}}$,
$g^{\mathbf{TA}/\leftrightarrow^*_{R_1}}([c]_{\leftrightarrow^*_{R_1}}, [c]_{\leftrightarrow^*_{R_1}}) = [a]_{\leftrightarrow^*_{R_1}}$.

Observe that $STY = COY$.

References

[1] W. S. Brainerd, Tree generating regular systems, *Information and Control* **14** (1969), 217-231.

[2] H. Comon, Complete axiomatizations of some quotient term algebras, *Theoretical Computer Science* **118** (1993) 167-191.

[3] H. Comon, M. Dauchet, R. Gilleron, C. Löding, F. Jacquemard, D. Lugiez, S. Tison, M. Tommasi, Tree Automata Techniques and Applications, http://www.grappa.univ-lille3.fr/tata, October, 12th 2007.

[4] H. Comon, G. Godoy, R. Nieuwenhuis, The Confluence of Ground Term Rewrite Systems is Decidable in Polynomial Time, *42nd Annual Symposium on Foundations of Computer Science, FOCS 2001*, IEEE Computer Society, 2001, 298-307.

[5] M. Dauchet, P. Heuillard, P. Lescanne and S. Tison, Decidability of the confluence of finite ground term rewrite systems and of other related term rewrite systems, *Information and Control* **88** (1990) 187-201.

[6] M. Dauchet and S. Tison, The Theory of Ground Rewrite Systems is Decidable, in: *Proc. 5th IEEE Symposium on Logic in Computer Science*, Philadelphia (1990) 242-248.

[7] J. Engelfriet, Derivation Trees of Ground Term Rewriting Systems, *Information and Computation* **152** 1999 1-15.

[8] Z. Fülöp, and S. Vágvölgyi, Congruential tree languages are the same as recognizable tree languages - A proof for a theorem of D. Kozen, *Bulletin of the EATCS*, **39** (1989) 175-185.

[9] Z. Fülöp, and S. Vágvölgyi, Ground term rewriting rules for the word problem of ground term equations, *Bulletin of the EATCS*, **45** (1991) 186-201.

[10] Z. Fülöp and S. Vágvölgyi, Minimal Equational Representations of Recognizable Tree Languages, *Acta Informatica*, **34** (1997) 59-84.

[11] Z. Fülöp and S. Vágvölgyi, Restricted Ground Tree Transducers, *Theoretical Computer Science*, 250 (2001) 219-233.

[12] J. Gallier, P. Narendran, D. Plaisted, S. Raatz, and W. Snyder, An Algorithm for Finding Canonical Sets of Ground Rewrite Rules in Polynomial Time, *Journal of the Association for Computing Machinery*, **40** (1993) 1-16.

[13] Gécseg F, Steinby M (1984) Tree Automata. Akadémiai Kiadó, Budapest

[14] D. Kapur, Shostak's Congruence Closure as Completion, Rewriting Techniques and Applications, Proc. of the 8th International Conference, (Sitges, Spain, 1997) *Lecture Notes in Computer Science*, Vol. 1232, Springer Publishing Company, New York, 1997, 23-37.

[15] D. Kozen, Complexity of finitely presented algebras, Proc. 9th Annual ACM Symp. on Theory of Computing (Boulder Colorado, 1977) 164-177.

[16] D. Kozen, Partial automata and finitely generated congruences: An extension of Nerode's theorem, Report DAIMI PB - 400, Computer Science Department of Aarhus University, Denmark, 1992.

[17] D. Plaisted and A. Sattler-Klein, Proof Lengths for Equational Completion, *Information and Computation*, **125** (1996) 154-170.

[18] W. Snyder, A Fast Algorithm for Generating Reduced Ground Rewriting Systems from a set of Ground Equations, *Journal of Symbolic Computation*, **15** (1993) 415-450.

[19] S. Vágvölgyi, A fast algorithm for constructing a tree automaton recognizing a congruential tree language, *Theoretical Computer Science*, **115** (1993) 391-399.

[20] S. Vágvölgyi, Simple and Minimal Ground Term Equation Systems, *Fundamenta Informaticae* **36** (1998) 367-413.

[21] S. Vágvölgyi, Congruential Complements of Ground Term Rewrite Systems, *Theoretical Computer Science*, **238** (2000) 247-274.

[22] S. Vágvölgyi, Intersection of finitely generated congruences over term algebra, *Theoretical Computer Science*, **300** (2003) 209-234.

[23] S. Vágvölgyi, Deterministic bottom-up tree transducers and ground term rewrite systems, *Theoretical Computer Science* **410** (2009) 2250-2278.

[24] R. Verma, A. Hayrapetyan, A new decidability technique for ground term rewriting systems with applications. *ACM Transactions on Computational Logic*, **6** (2005) 102-123.